3D Hyperbolic Tiling and Horosphere Cross Section

Vladimir Bulatov,

Shapeways

Joint AMS/MAA meeting

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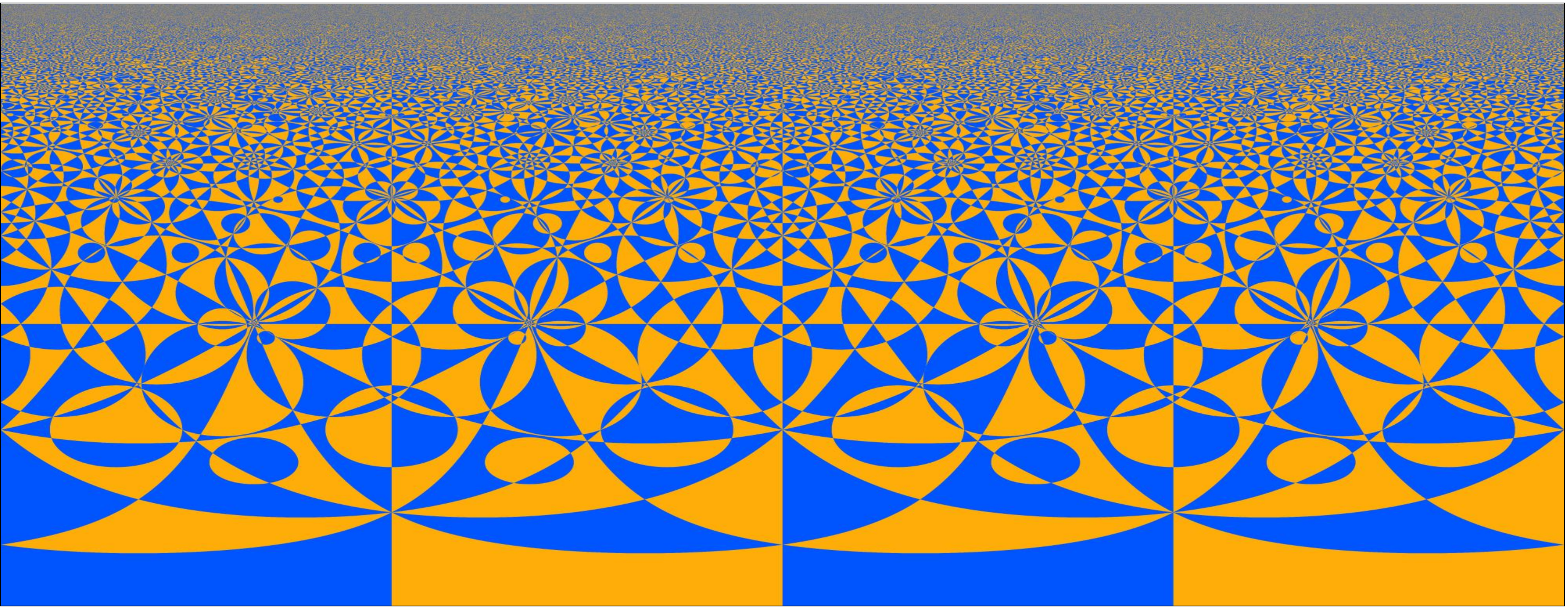
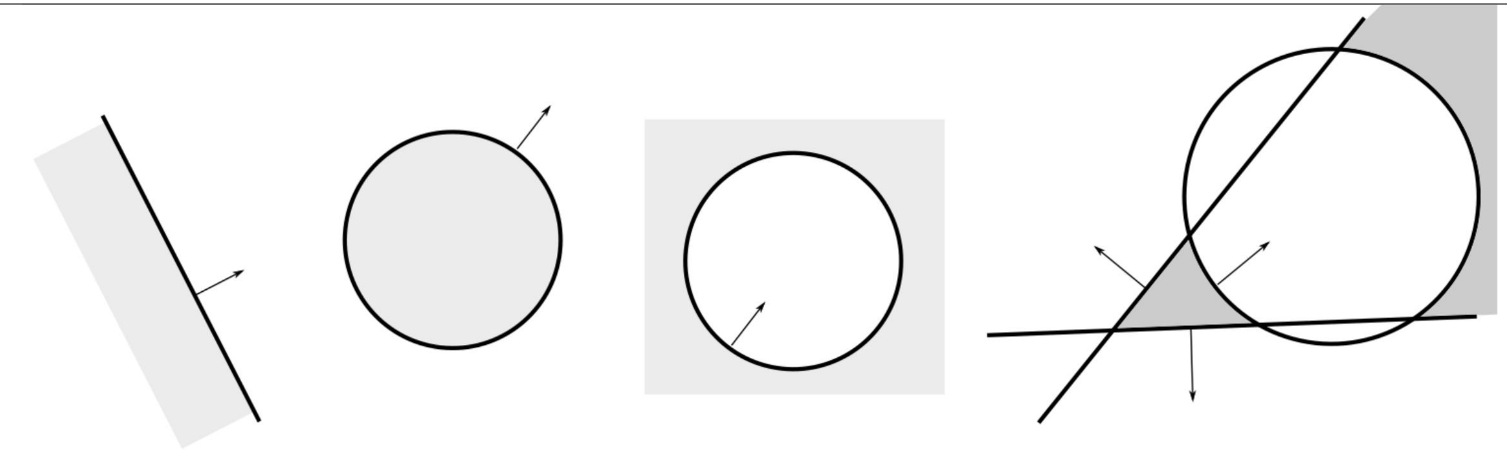


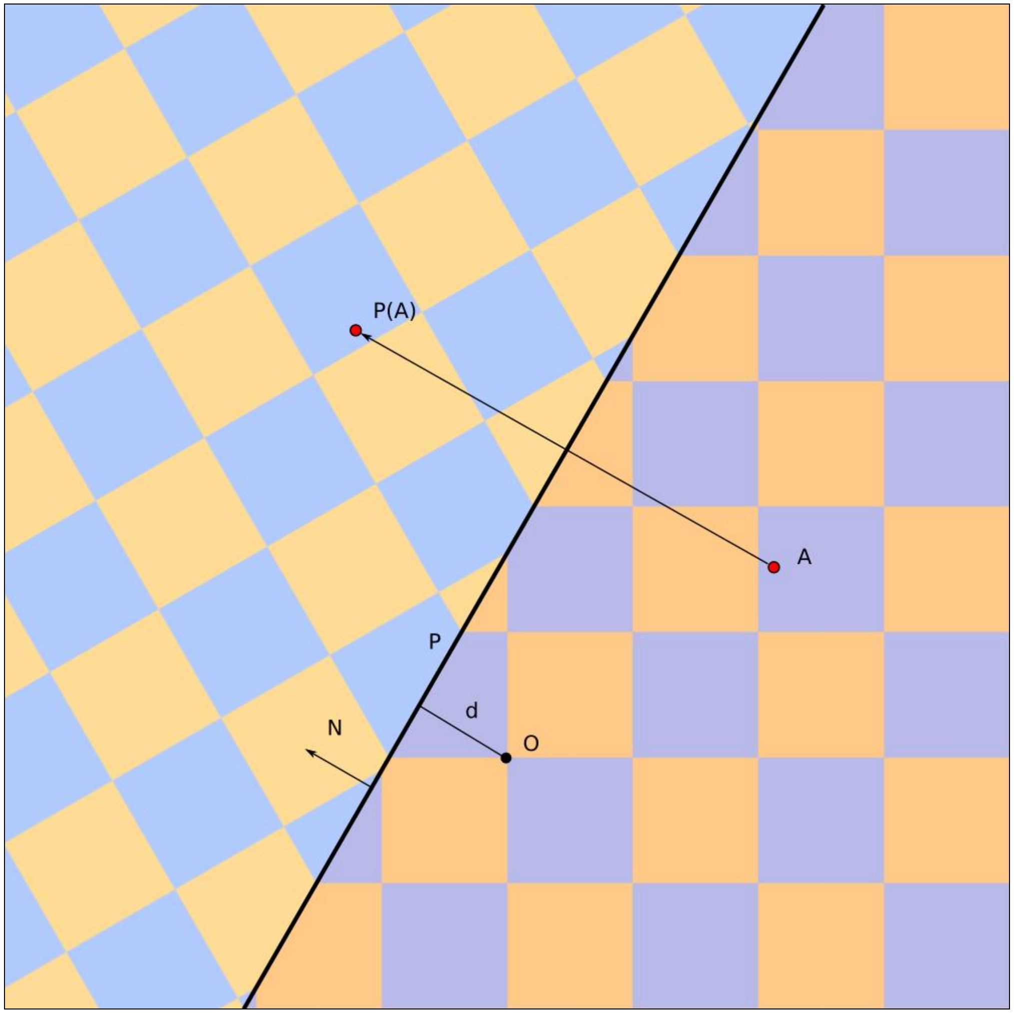
Inversive Geometry

- Convenient container to work with 3 dimensional hyperbolic tilings is 3D Inversive (or Moebius) Geometry. It can contain upper half space model and ball model of H^4.
- Geometric transformations are reflections in planes and inversions in spheres (splanes) and arbitrary compositions.
- Each splane divide space into two sets *interior* and *exterior*. Specific orientation is assigned to splanes by direction of normal that points to the **exterior** of the splane.
- Regions are defined as intersection of interiors of splanes.



Region on the right looks like two components region, however it is an infinite cheese wedge with a spherical hole.

Reflection in planes



Plane $P\{N,d\}$ is defined via N - vector of **external** normal and d - distance of plane to origin O.

Reflection in plane P maps point A into point $P(A)$

$P(A) = A - 2\langle A - dN, N\rangle N$.

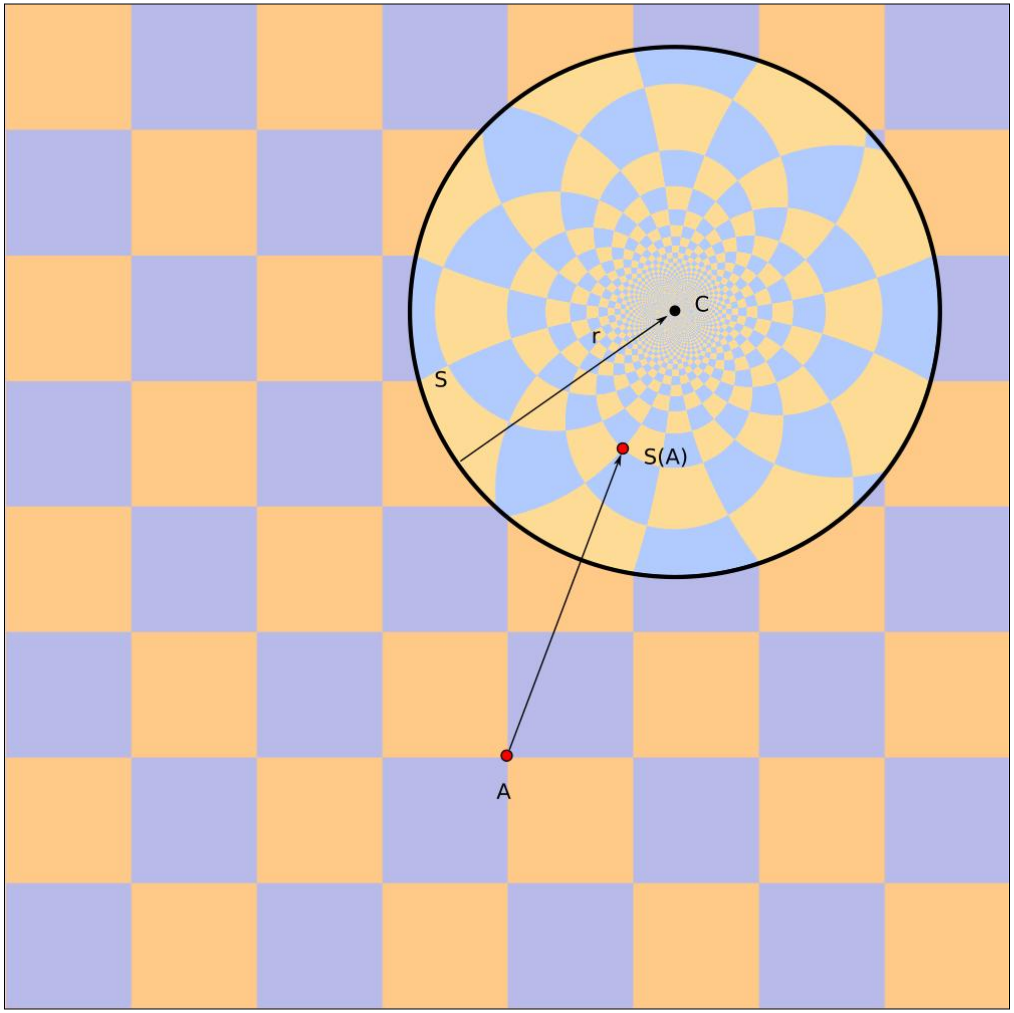
Where $\langle\, ,\, \rangle$ is scalar product.

Signed distance of point A to to plane P is

$P(S,A) = \langle A, N\rangle - d$

Here we have mapped interior region of the plane (on the right) on the exterior region on the left.

Inversion in spheres



Sphere $S\{C, r\}$ is defined via center C and signed radius r.

Positive radius means *interior* of splane S is region inside of the ball.

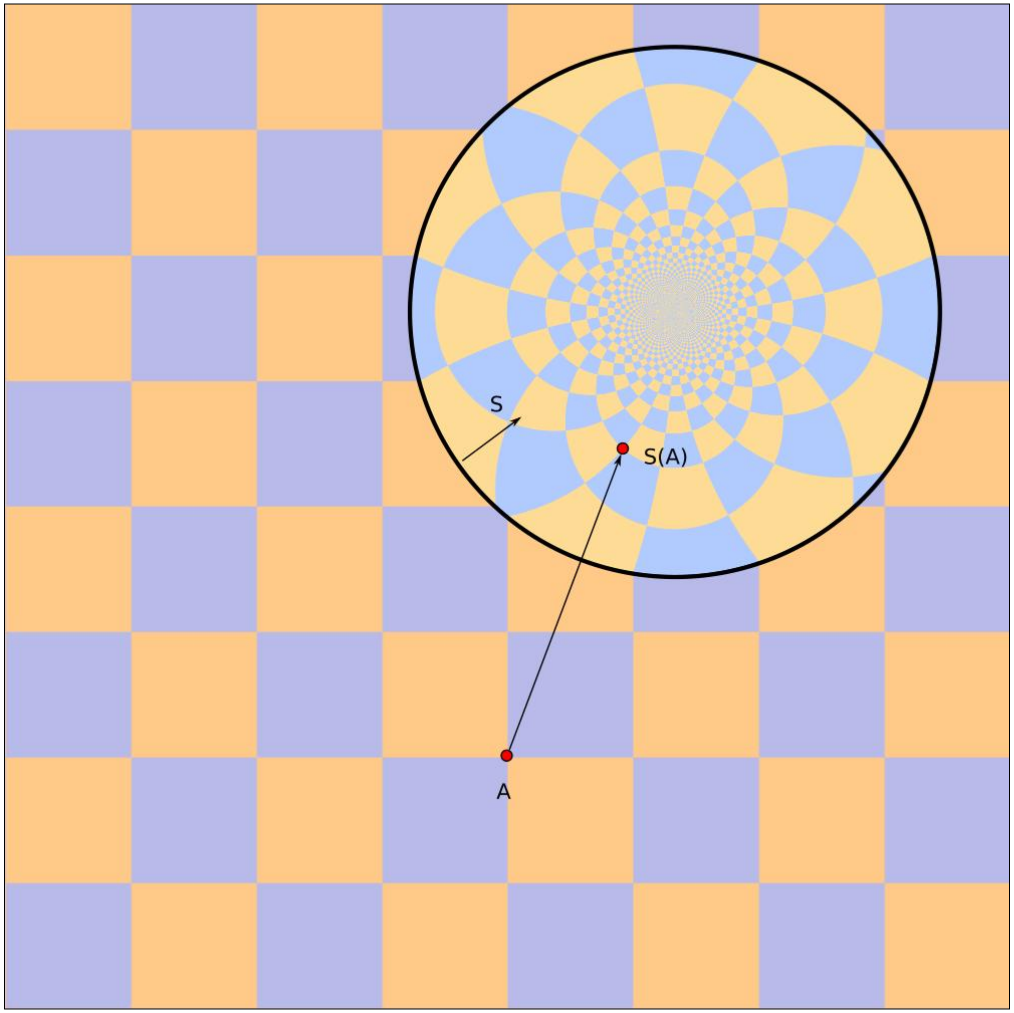
Negative radius means *interior* of splane is region outside of the ball.

Inversion in sphere S maps point A into point $S(A)$

$$S(A) = C + \frac{r^2}{|A - C|^2}(A - C)$$

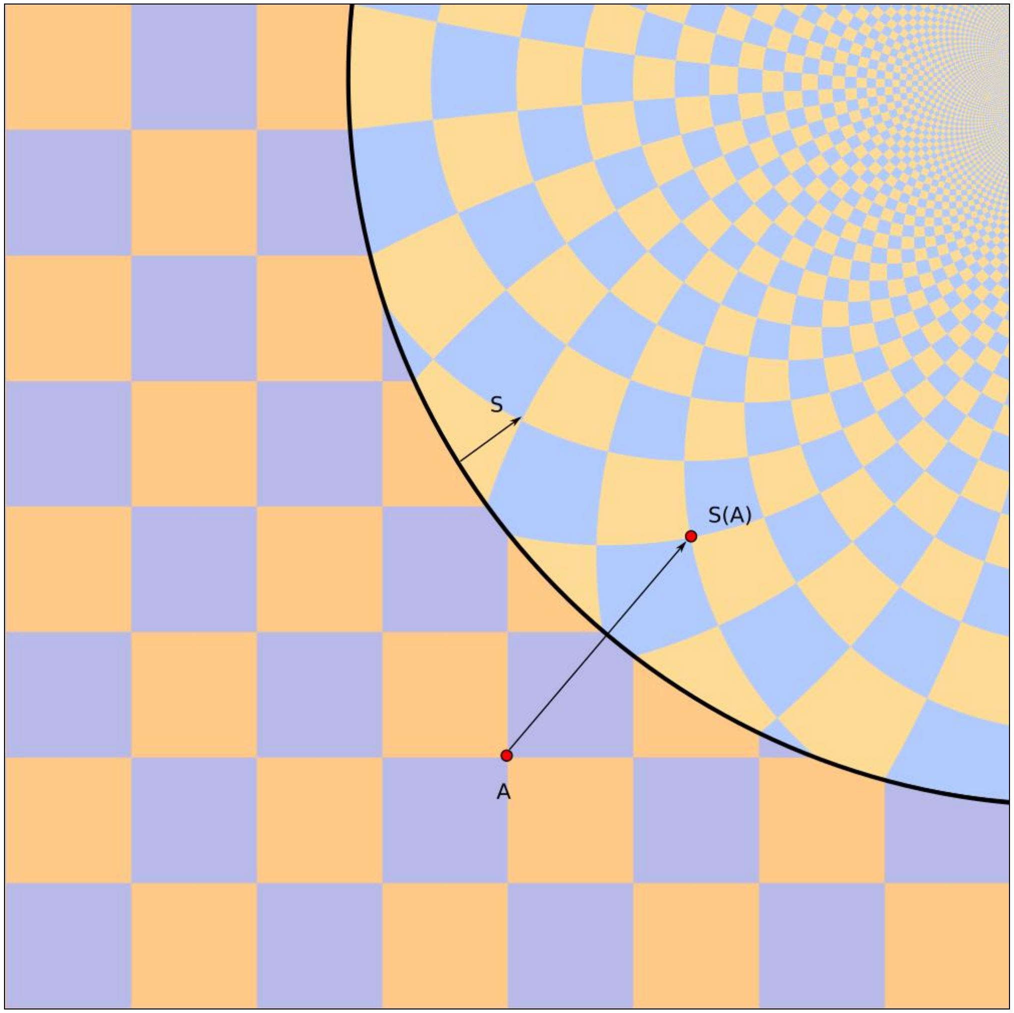
Signed distance of point A to to sphere S is $D(S, A) = sign(r)(|A - C| - |r|)$

Planes are spheres



Inversion is sphere becomes reflection in plane when radius of sphere tends to infinity.

Planes are spheres



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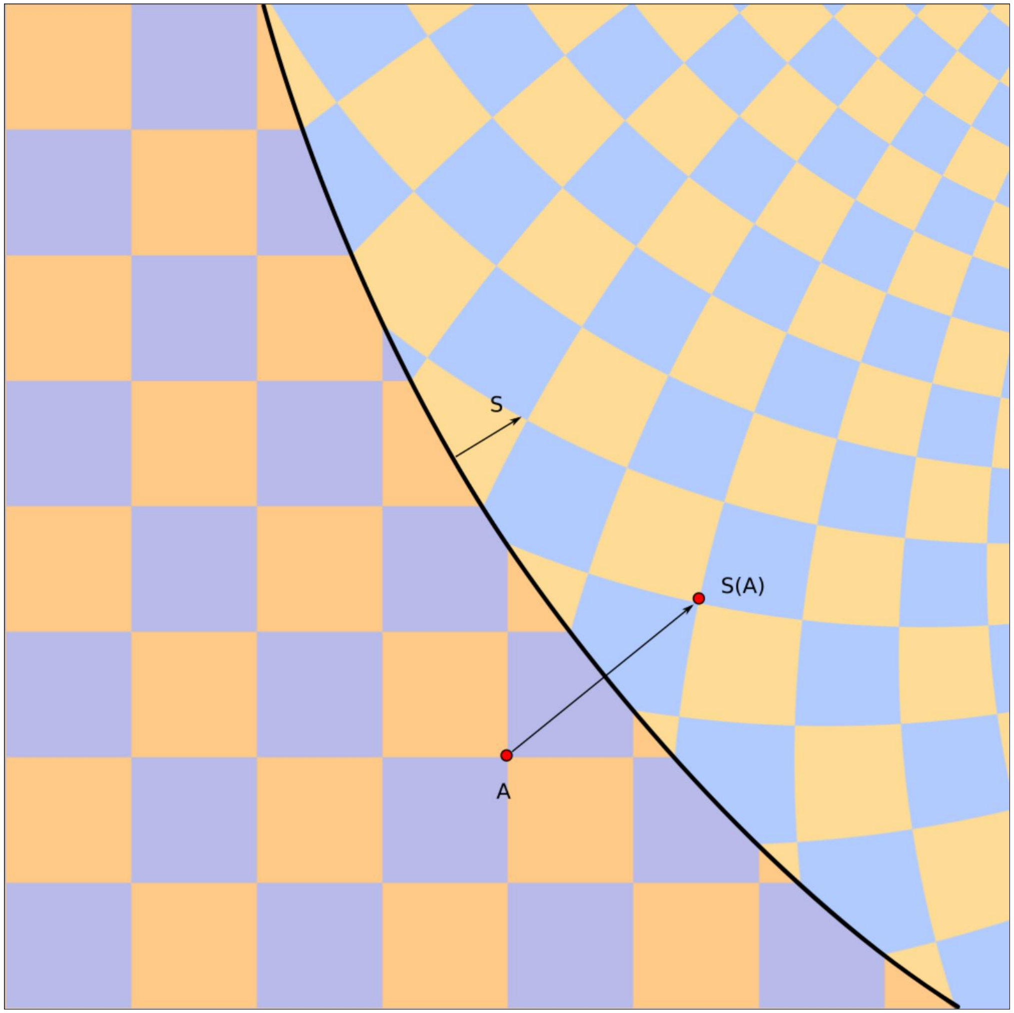

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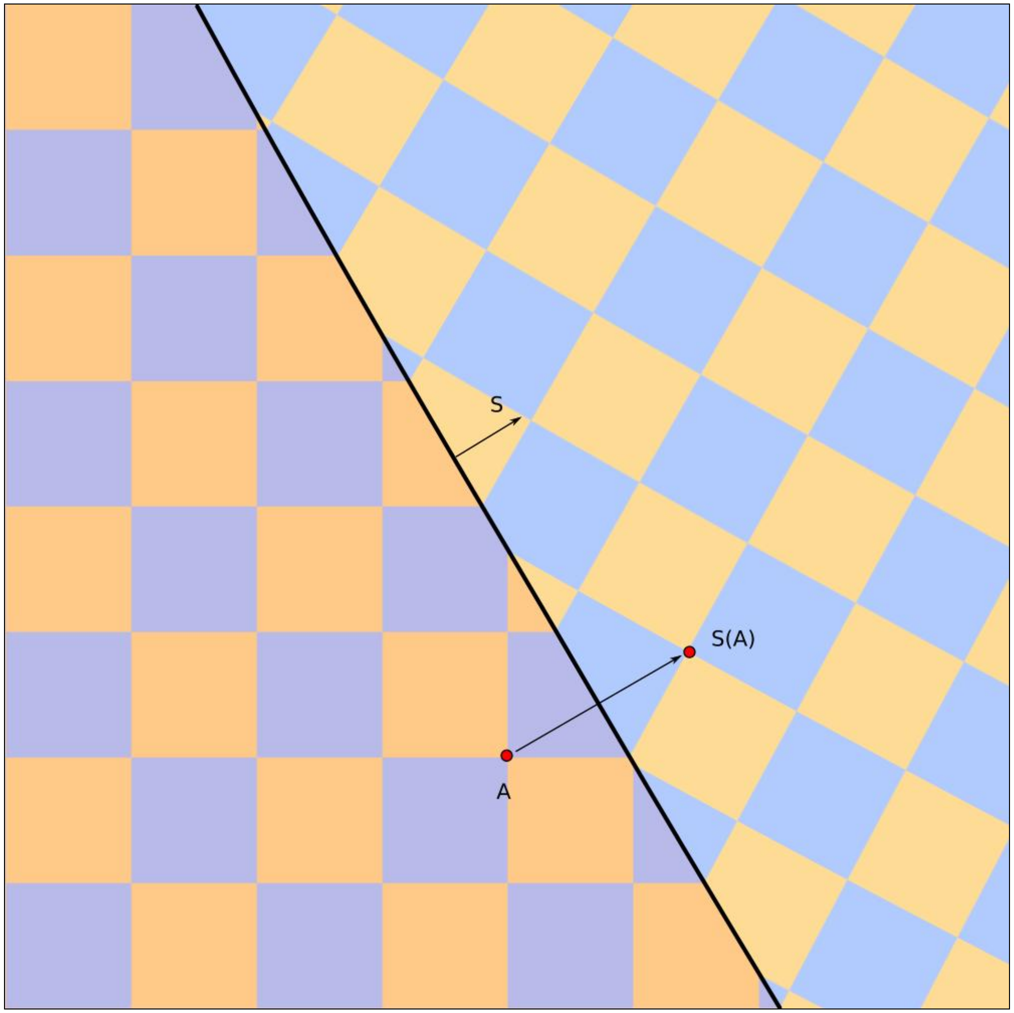
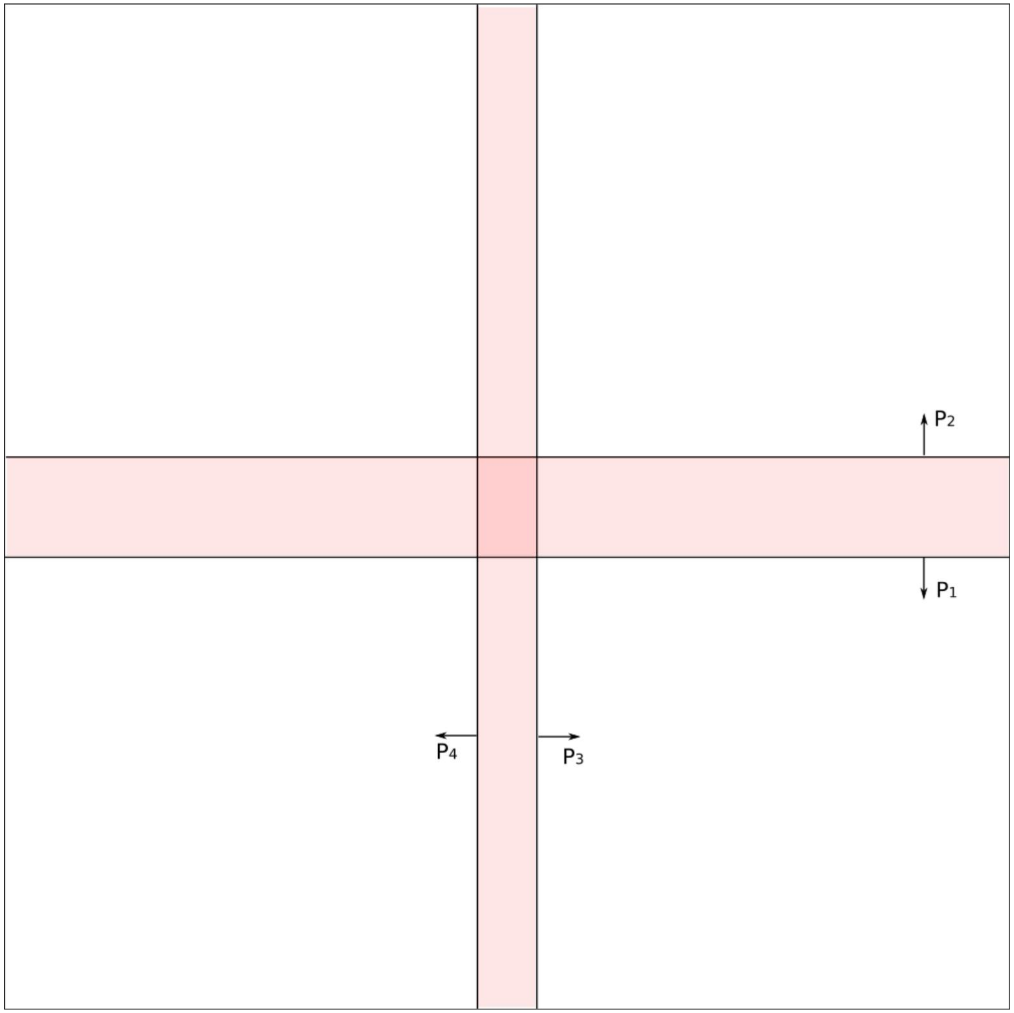
Planes are spheres



Inversion is sphere becomes reflection in plane when radius of sphere tends to infinity.

How to construct a tiling



Tiling can be constructed in several steps

- Build **Fundamental Domain** - region bounded by finite number of splanes (P_1, P_2, P_3, P_3)
- Dihedral angles between splanes are integral fraction of π: $a_{ij} = \frac{\pi}{m_{ij}}$
- Build symmetry group generated by reflections in splanes

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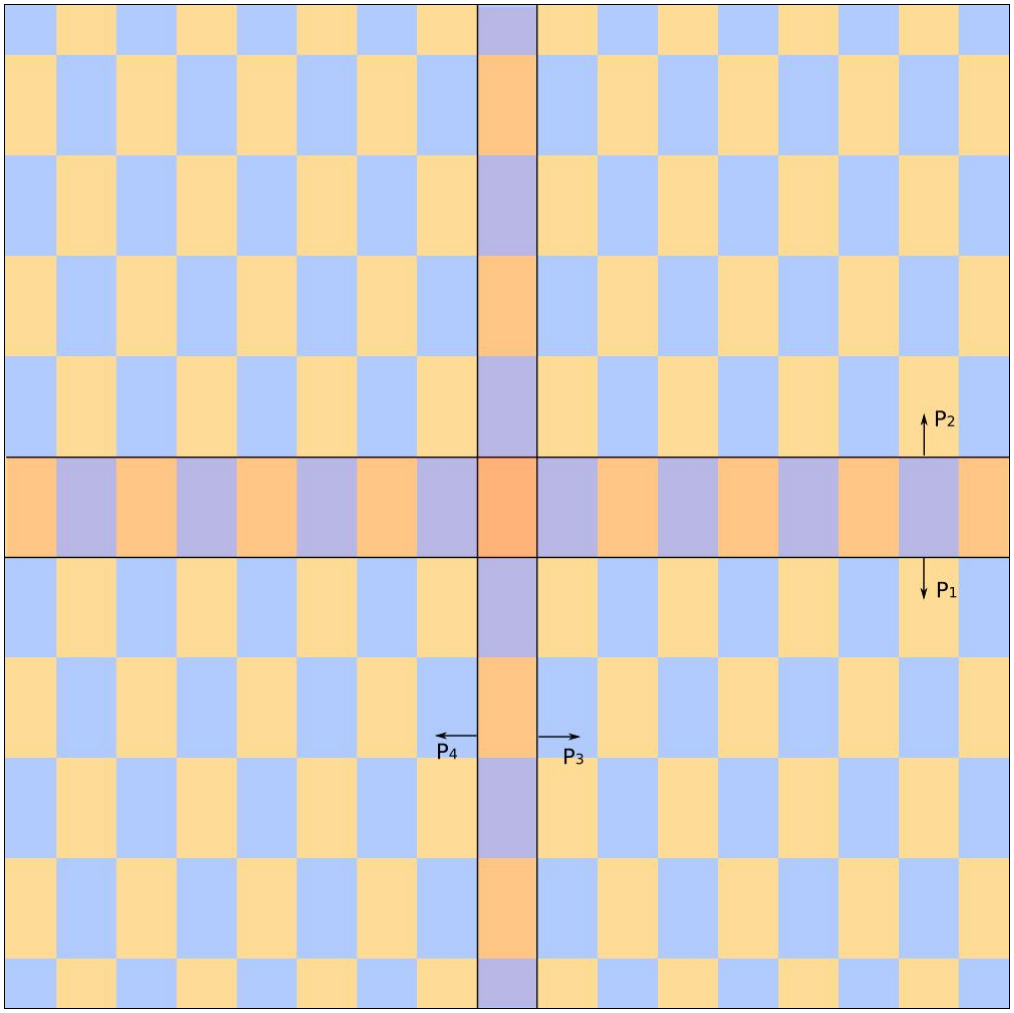

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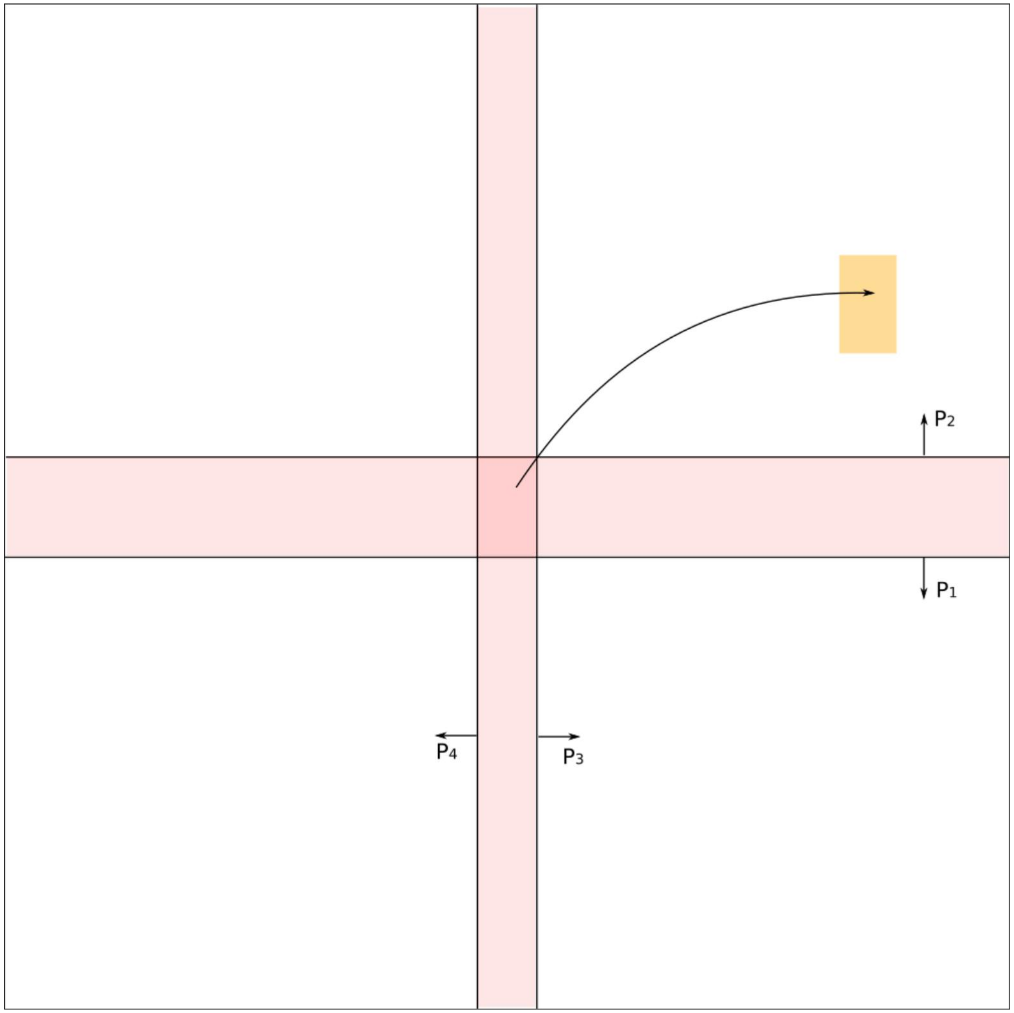
- Dihedral angles between splanes are integral fraction of π: $a_{ij} = \frac{\pi}{m_{ij}}$
- Build symmetry group generated by reflections in splanes
- Fill space by copies of Fundamental Domain transformed by all transformations of the group.

How to draw the tiling

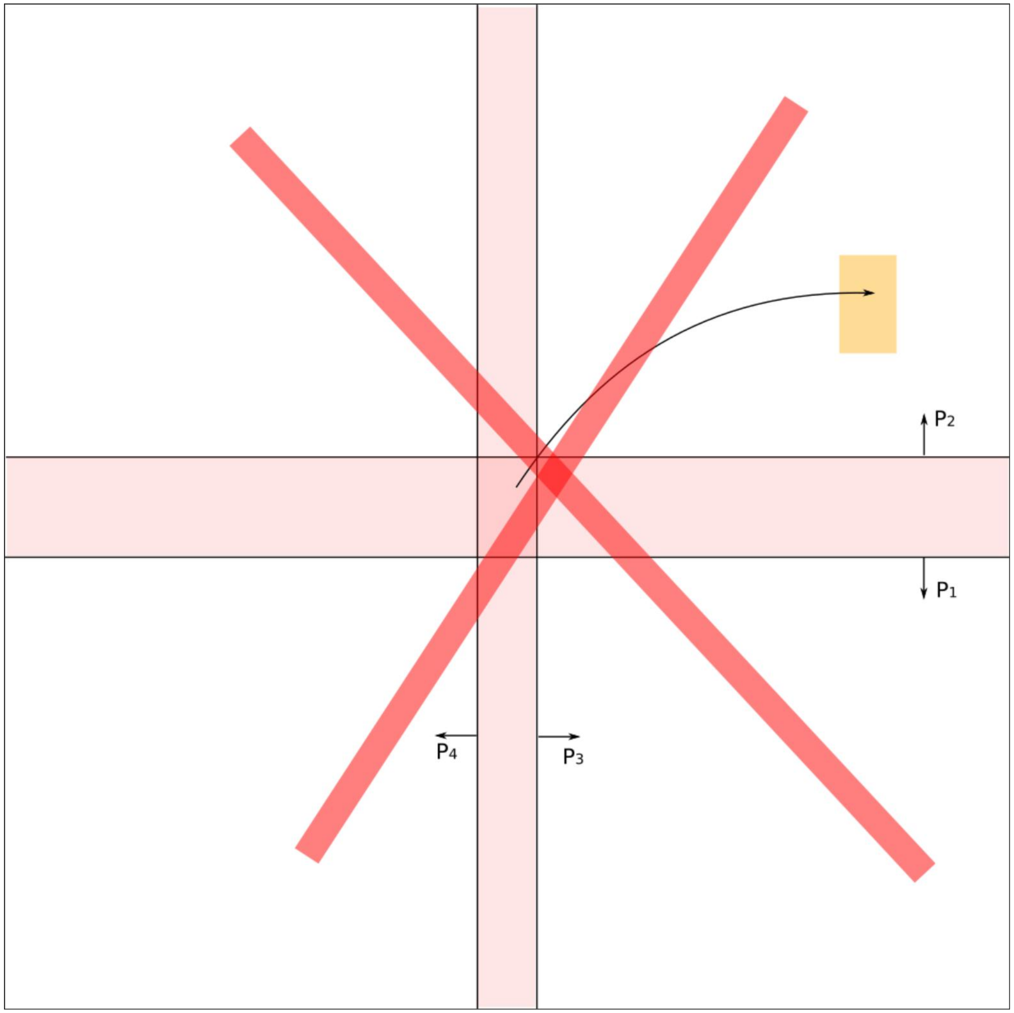


Straightforward appoach - apply each of transformation of the group to the fundamental domain.

Troubles

- Groups are usually infinite
- Fundamental Domain often is infinite
- Transformations of the group may act on the fundamental domain in rather complex way, mapping finite regions into infinity.

How to draw the tiling



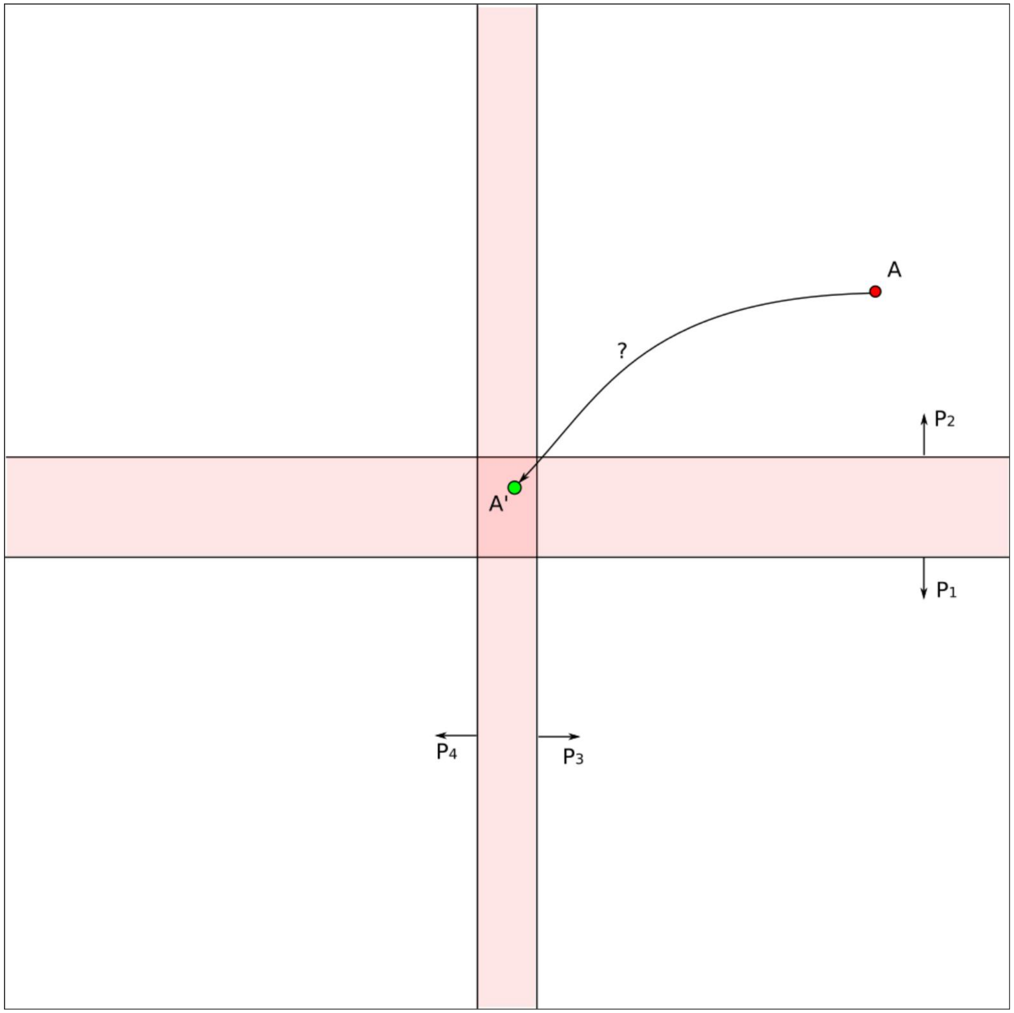
Straightforward appoach - apply each of transformation of the group to the fundamental domain.

Troubles

- Groups are usually infinite
- Fundamental Domain often is infinite
- Transformations of the group may act on the fundamental domain in rather complex way, mapping finite regions into infinity.

Different apporach is needed!

Reverse Pixels Mapping

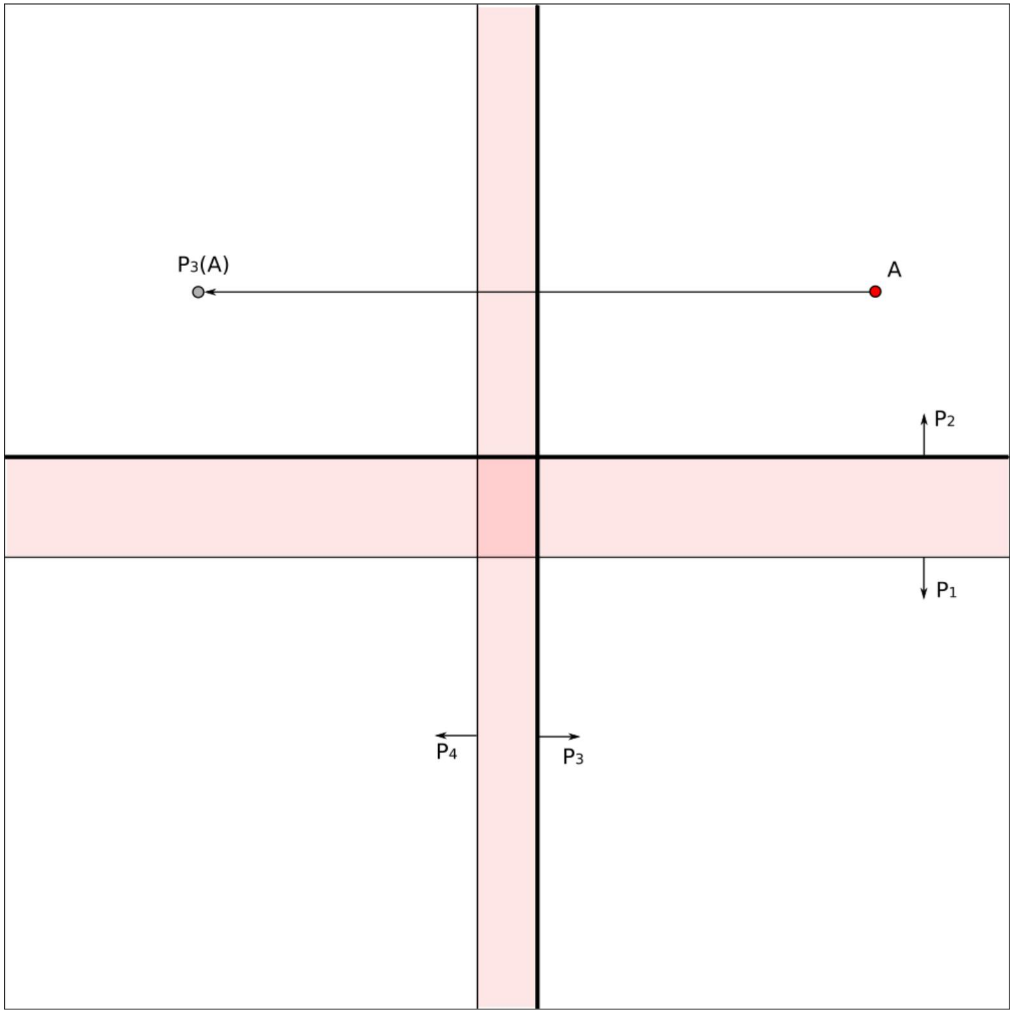


Different approach - draw only pixels which we can see.

For each visible pixel on the screen find transformation which maps it to the corresponding pixel in the fundamental domain.

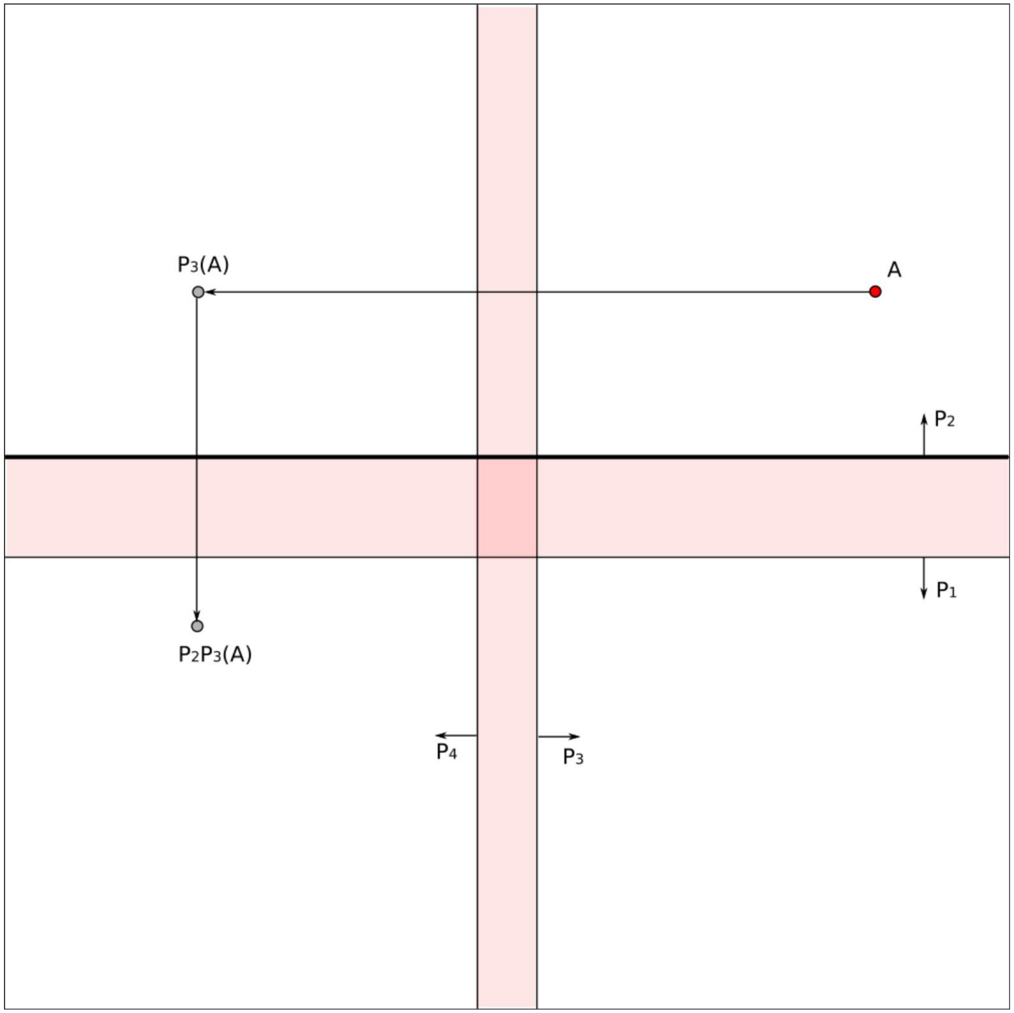
Color the pixel with color of the corresponding pixel in the fundamentral domain.

Search of Reverse Mapping



- Check the point location relative to each generator.
- Reflect point in first found generator which has point in its exterior

Search of Reverse Mapping



- Check the point location relative to each generator.
- Reflect point in first found generator which has point in its exterior
- Continue the process

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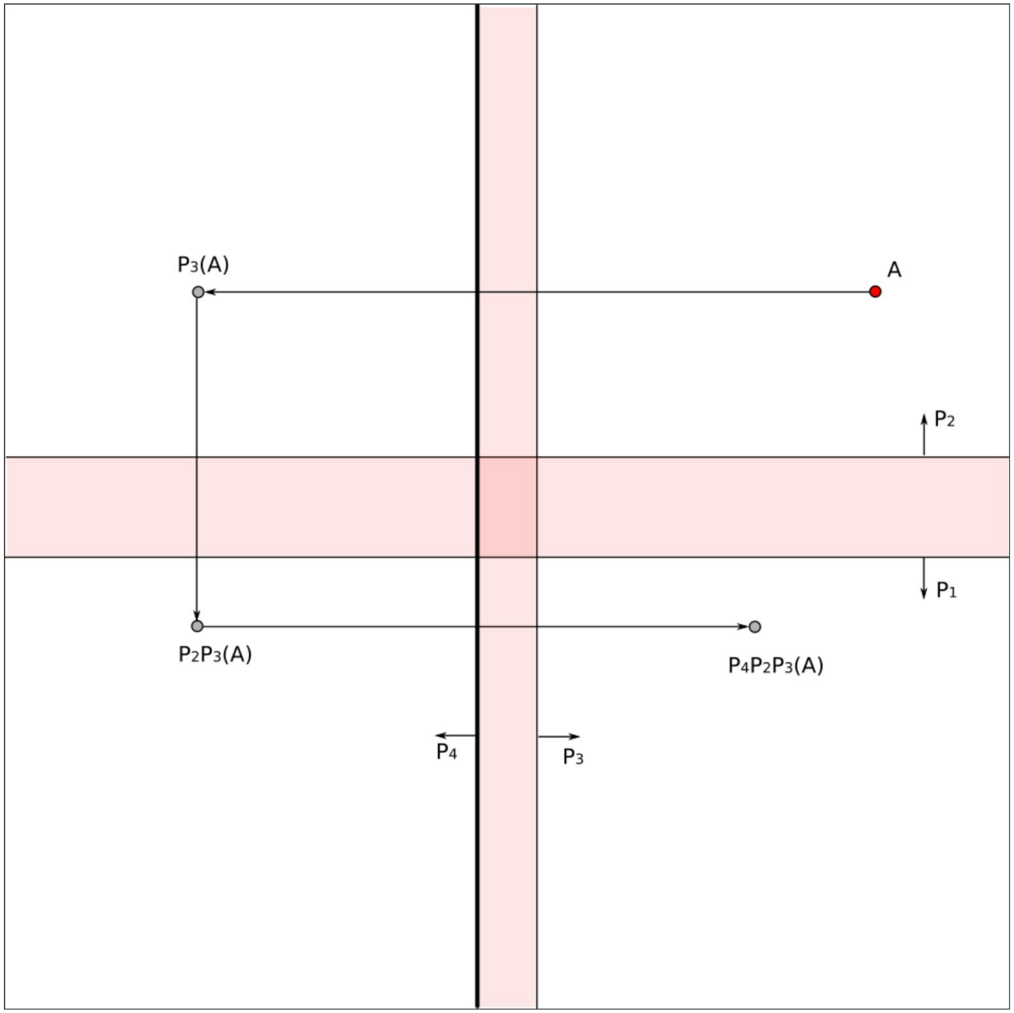

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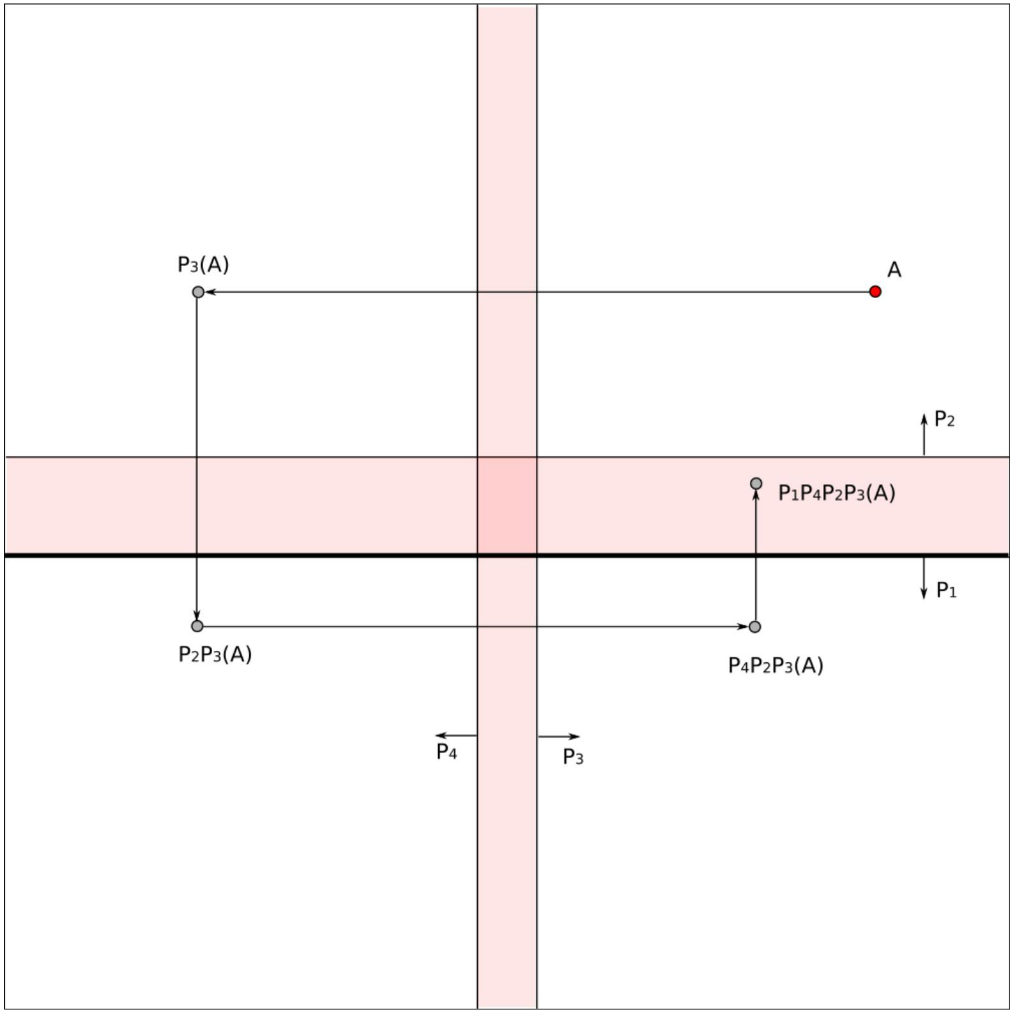

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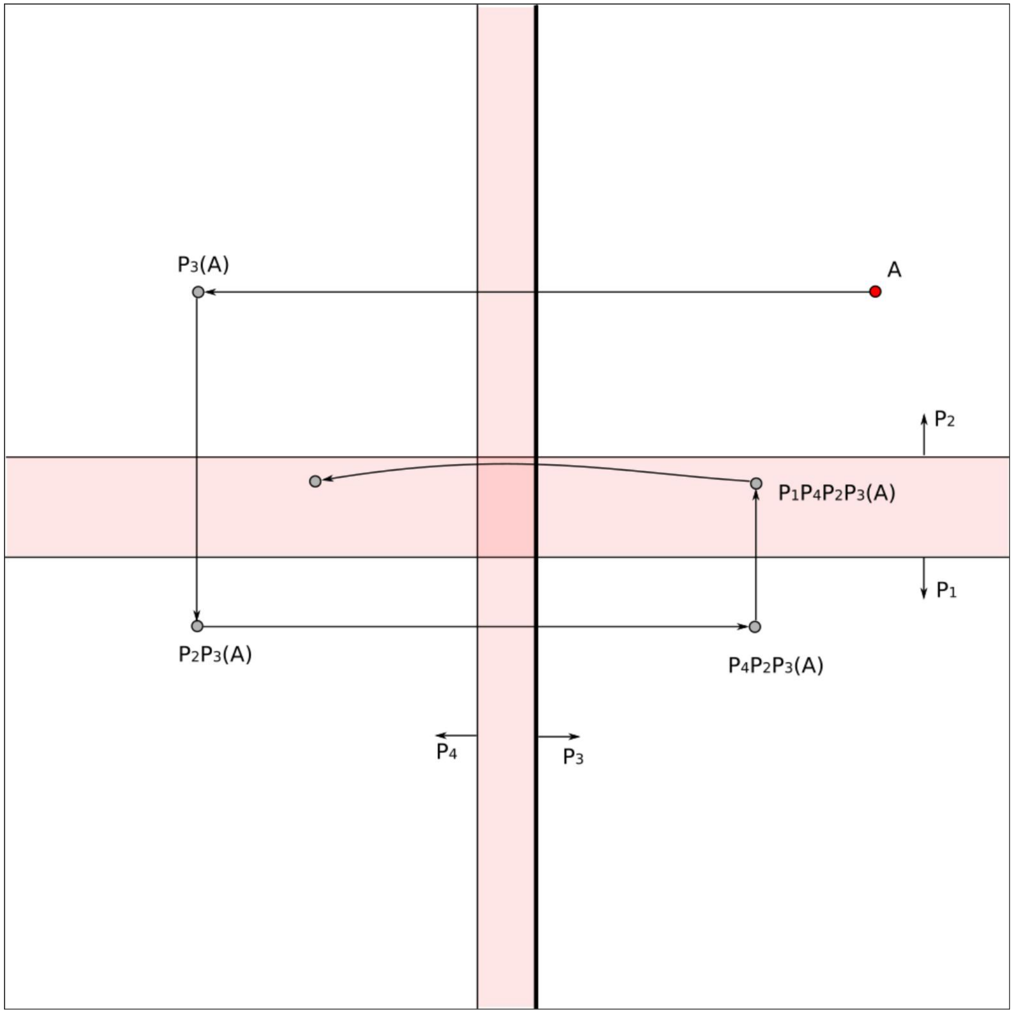

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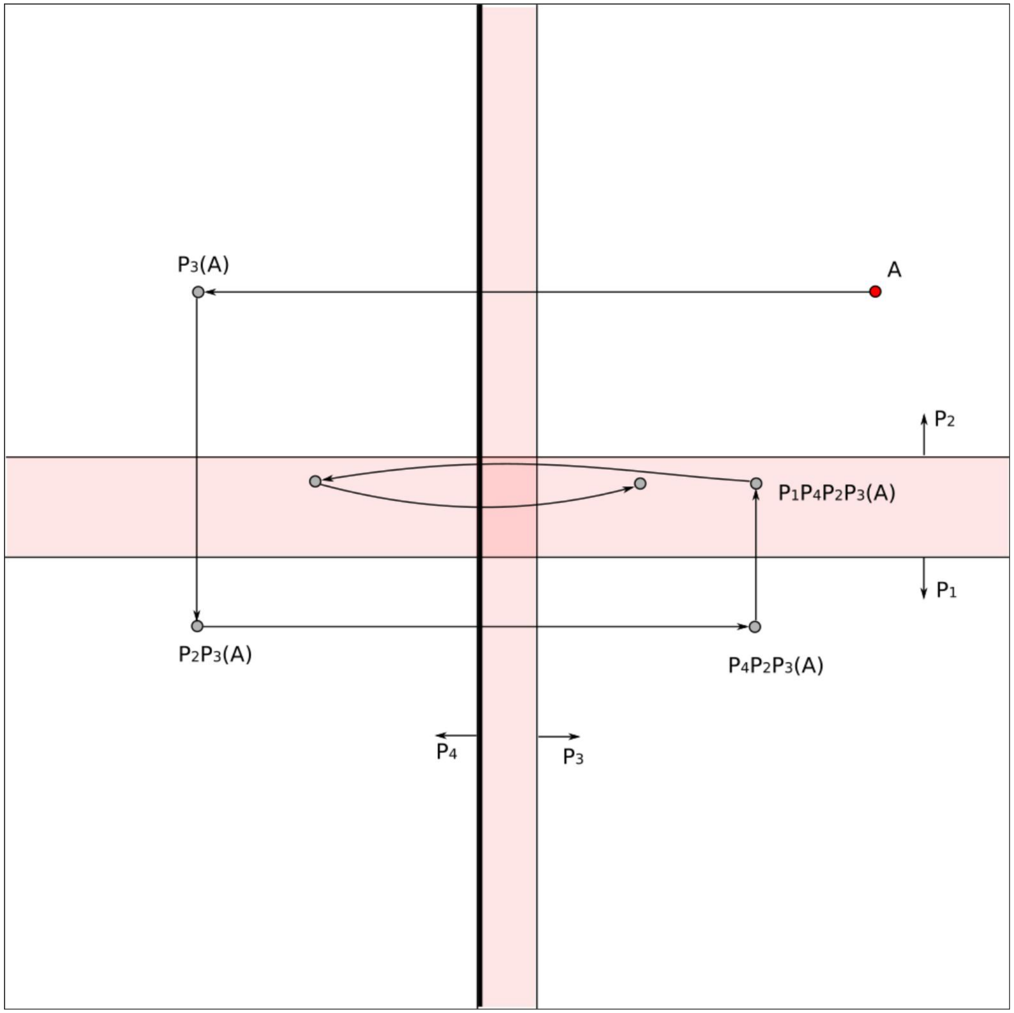

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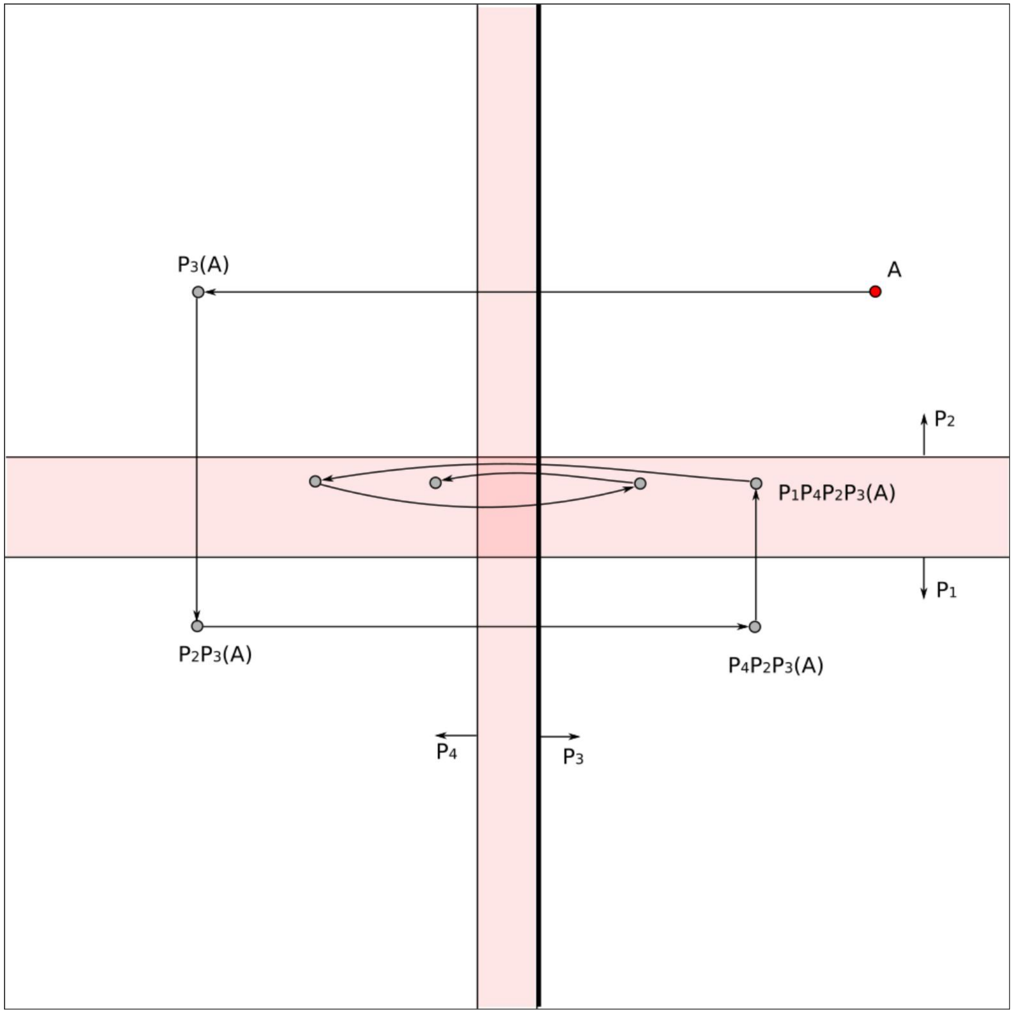
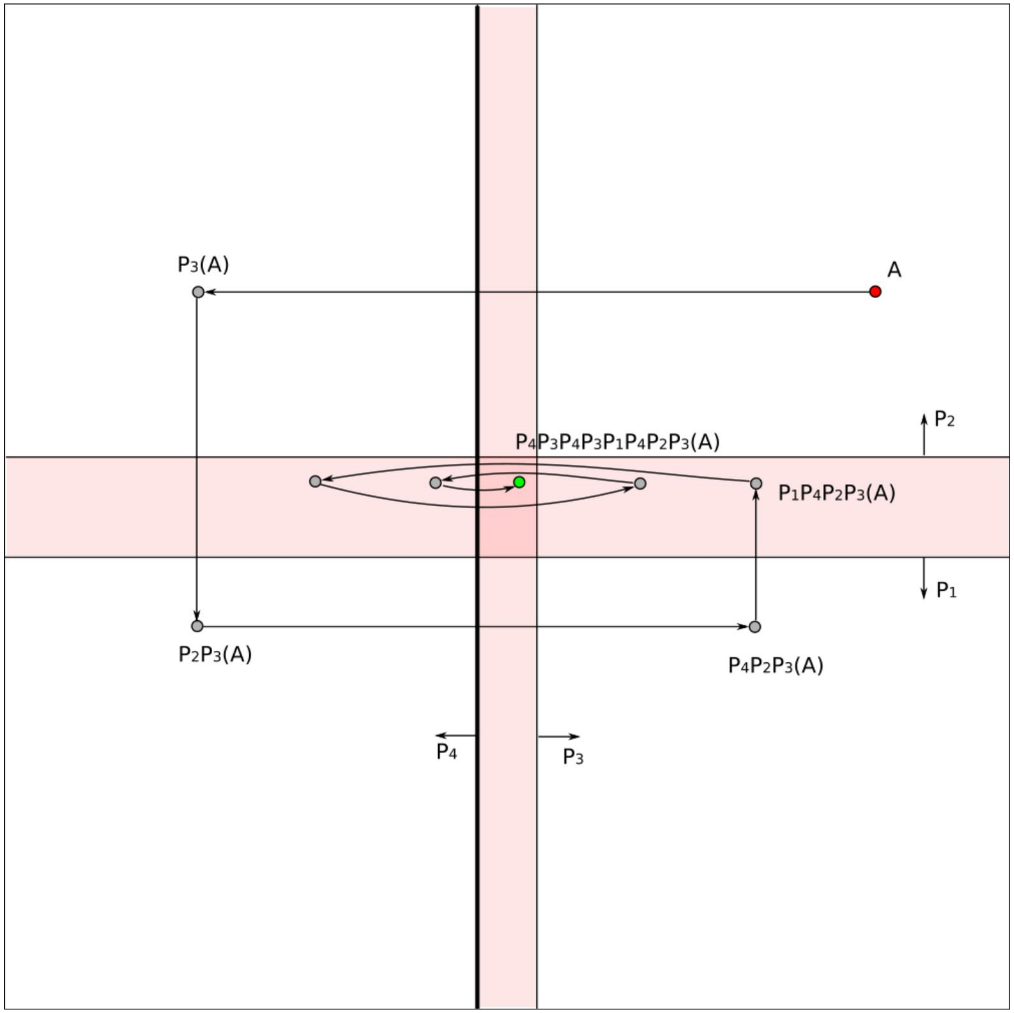


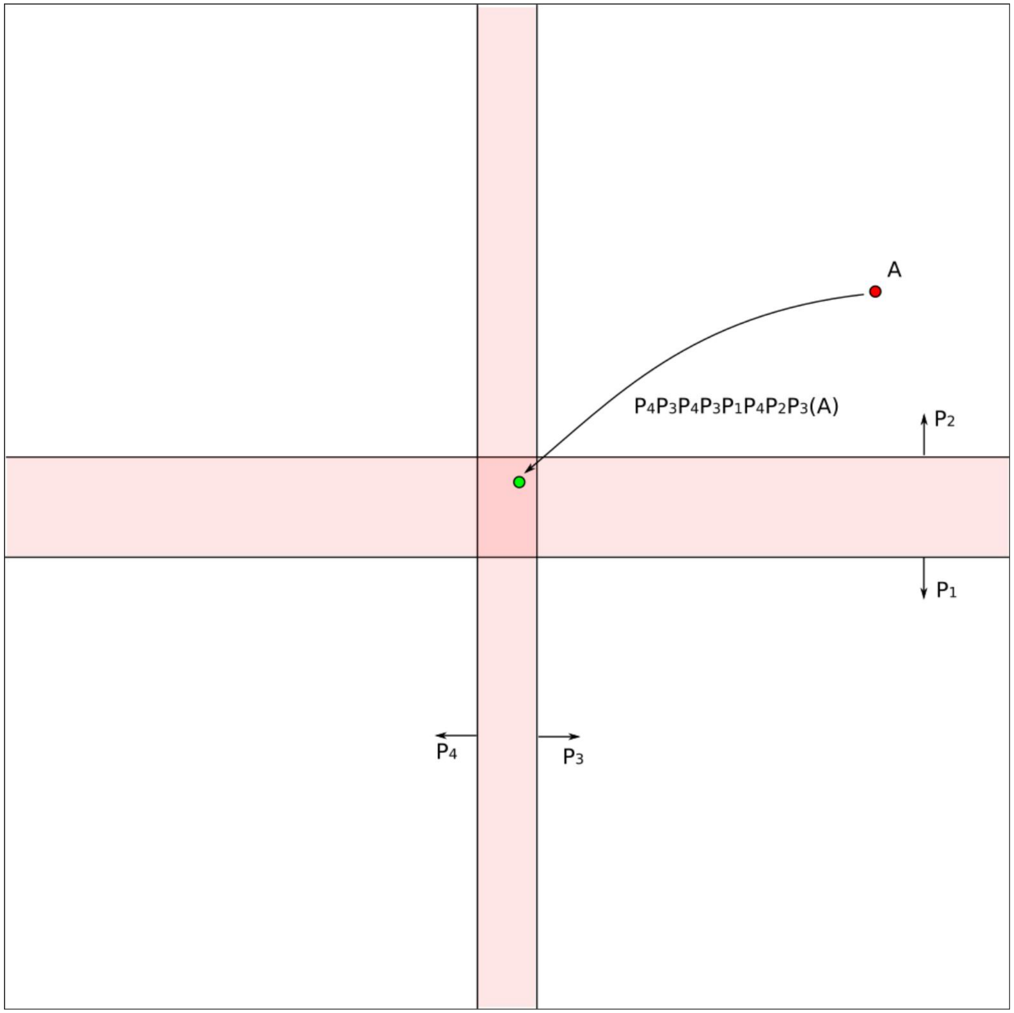
- Check the point location relative to each generator.
- Reflect point in first found generator which has point in its exterior
- Continue the process

Search of Reverse Mapping



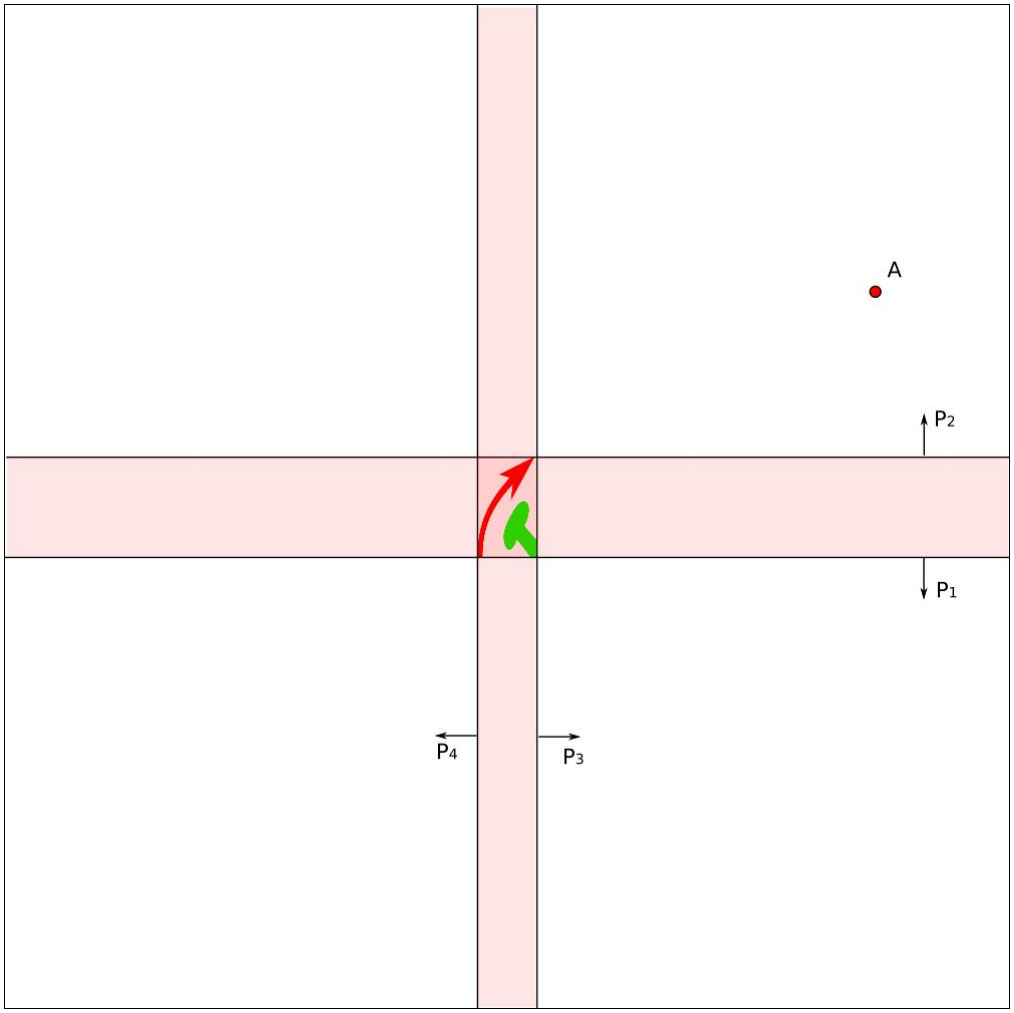
- Check the point location relative to each generator.
- Reflect point in first found generator which has point in its exterior
- Continue the process until the point is **inside** of each generator.

Search of Reverse Mapping



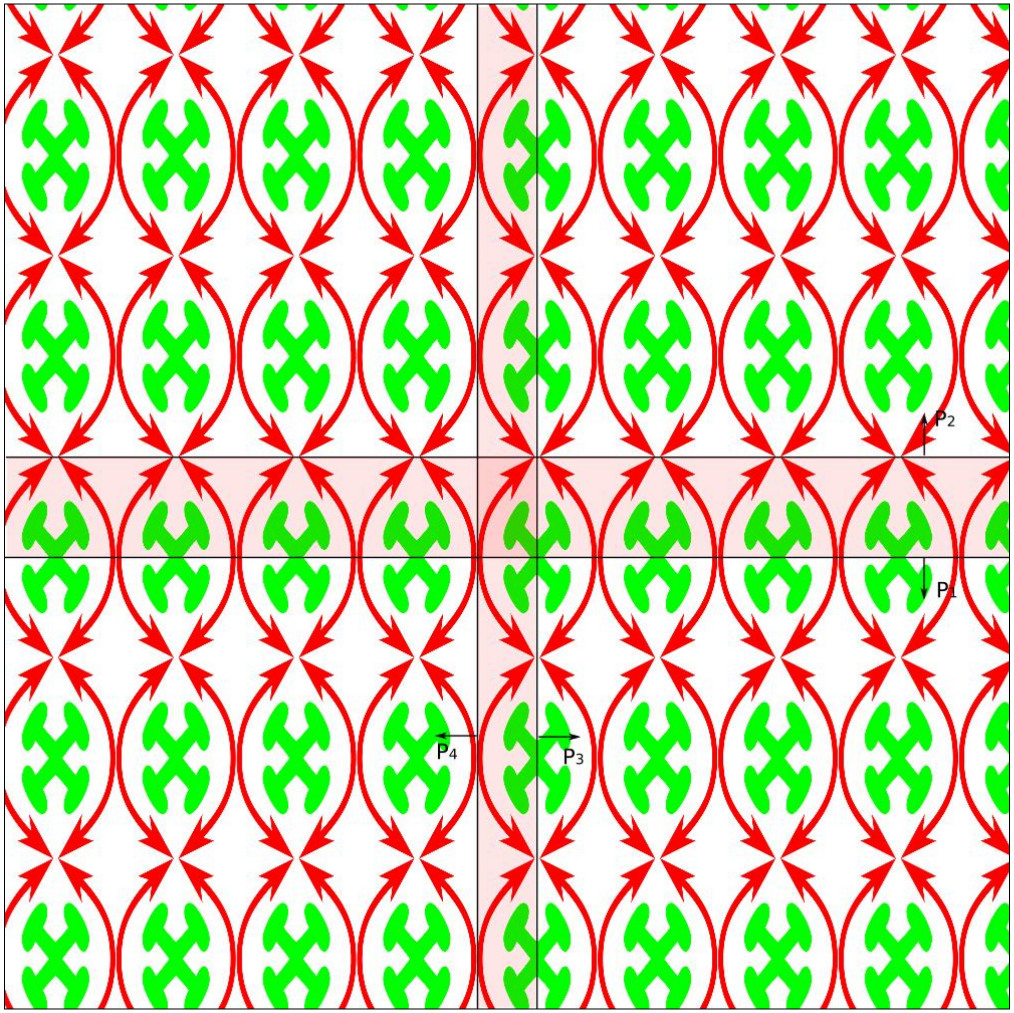
The composition of reflections gives the required transformation

Coloring the Pixels



Arbitrary pattern can be placed inside of the fundamental domain

Coloring the Pixels

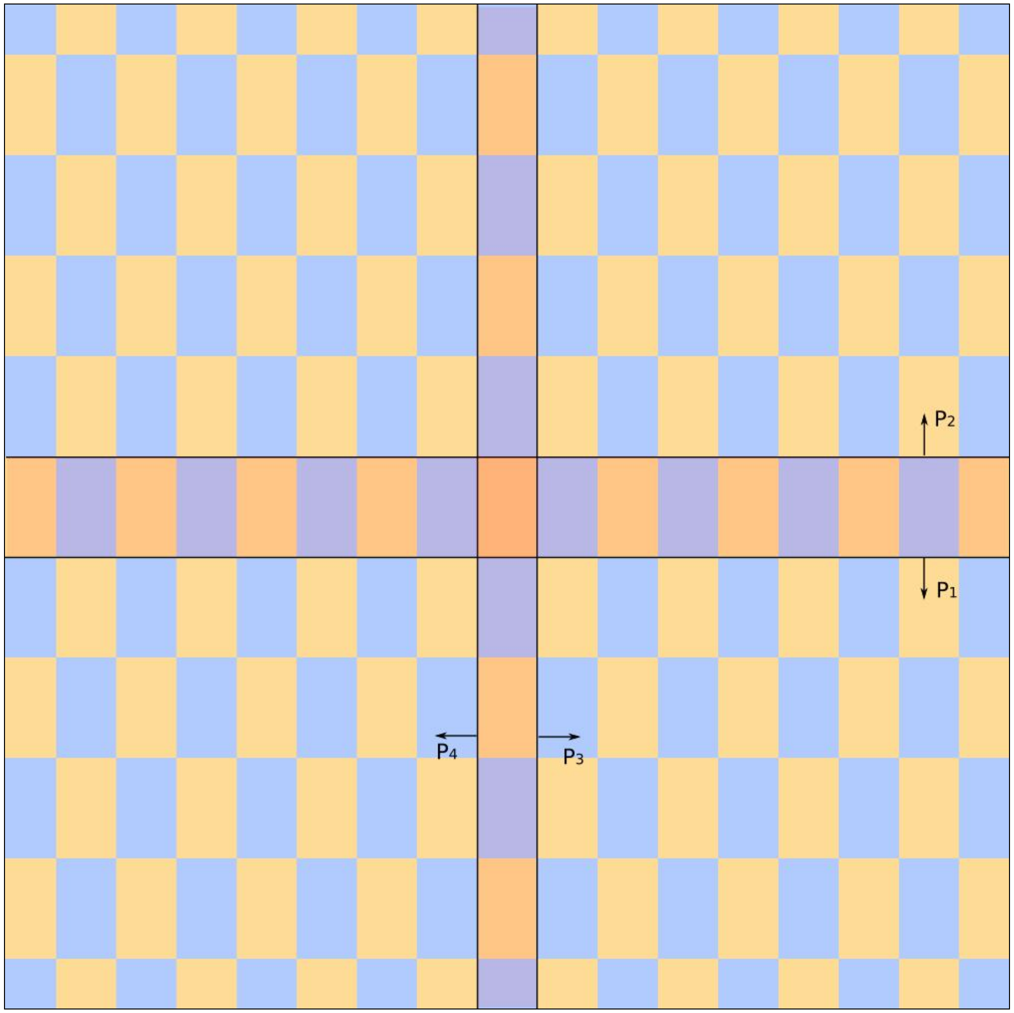


Arbitrary pattern can be placed inside of the fundamental domain

The pattern is reflected to the whole **visible** space

The calculations for each pixel are independent and can be run on GPU in paralel in real time

Coset coloring

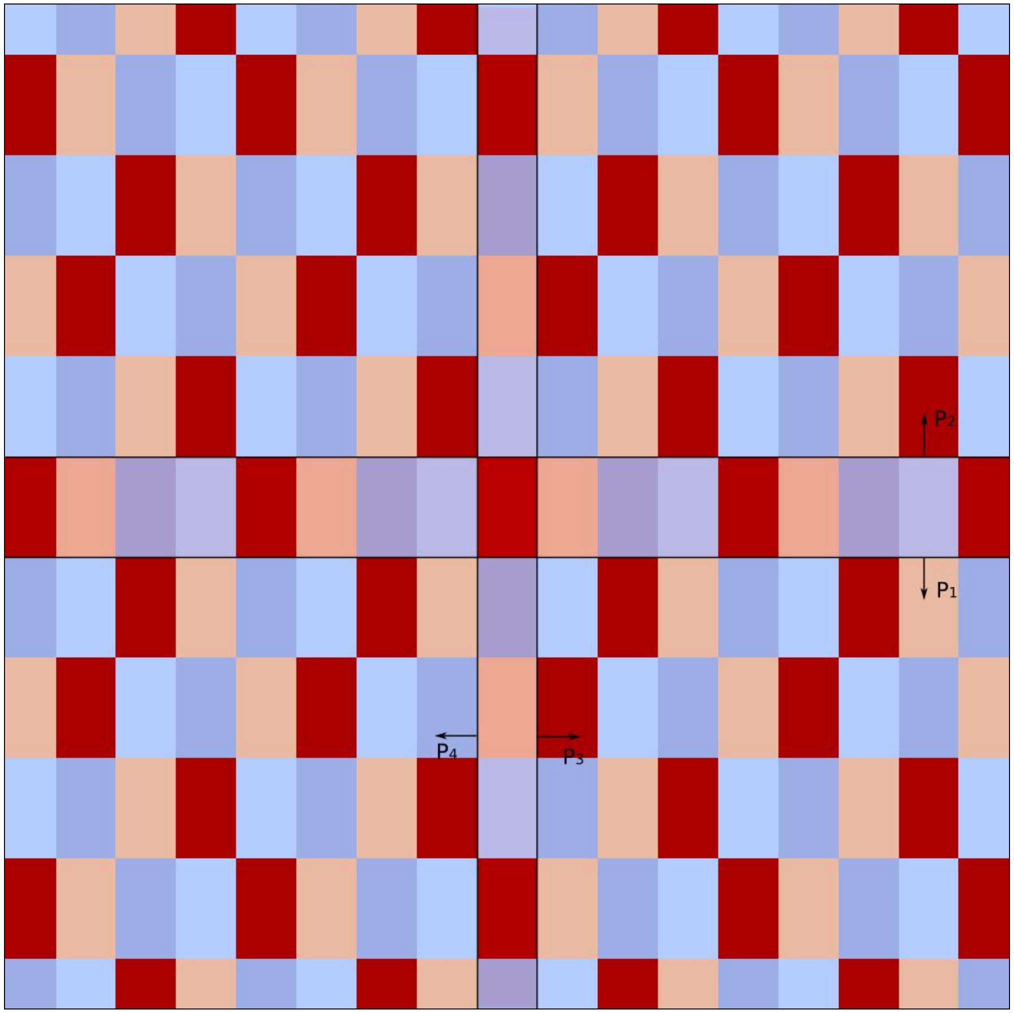


Another way to color the pixel is **coset coloring**.

Here pixels are assigned different color according to coset of the orientation preserving subgroup of index 2.

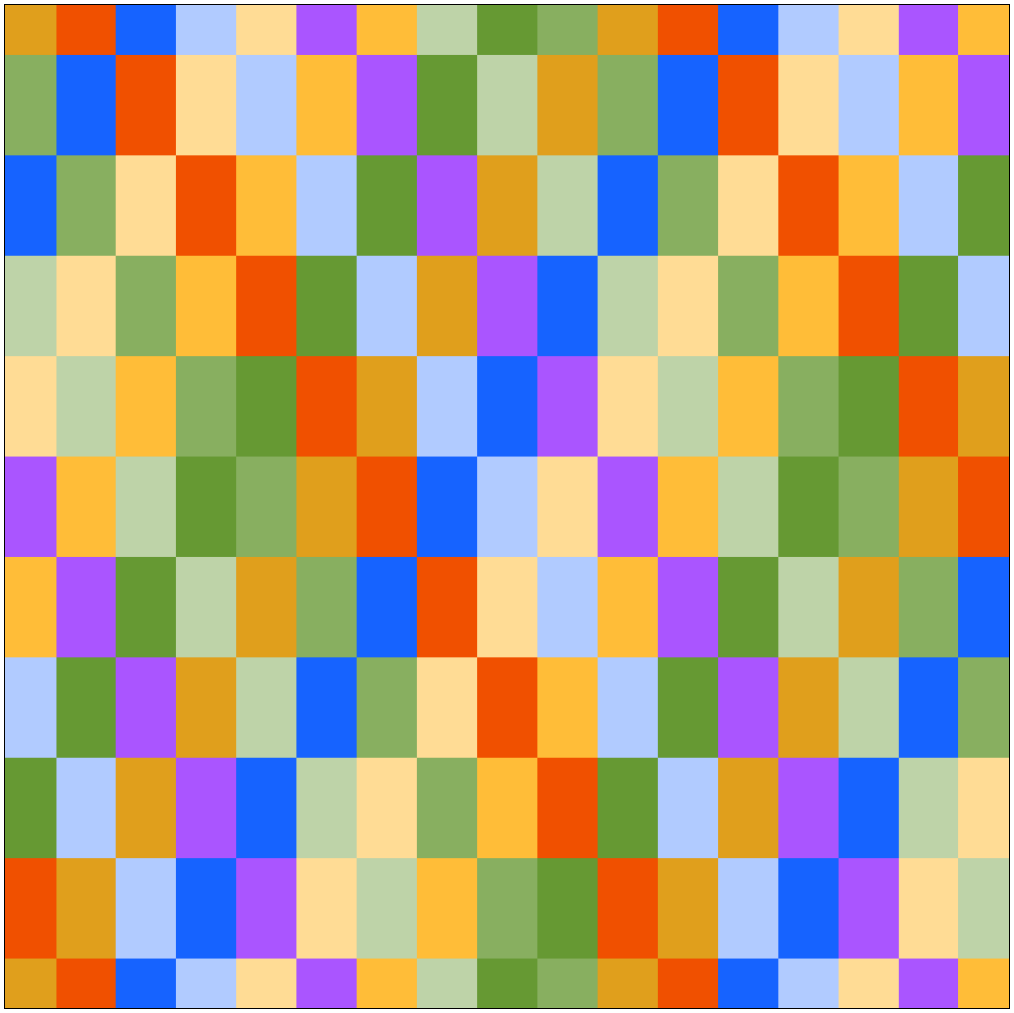
In this simplest case coset coloring can be obtained by counting number of reflections needed to map pixel to fundamental domain and assigning one color if count is even and another color if count is odd.

Coset coloring



Color count equals to the index of subgroup.

Coset coloring

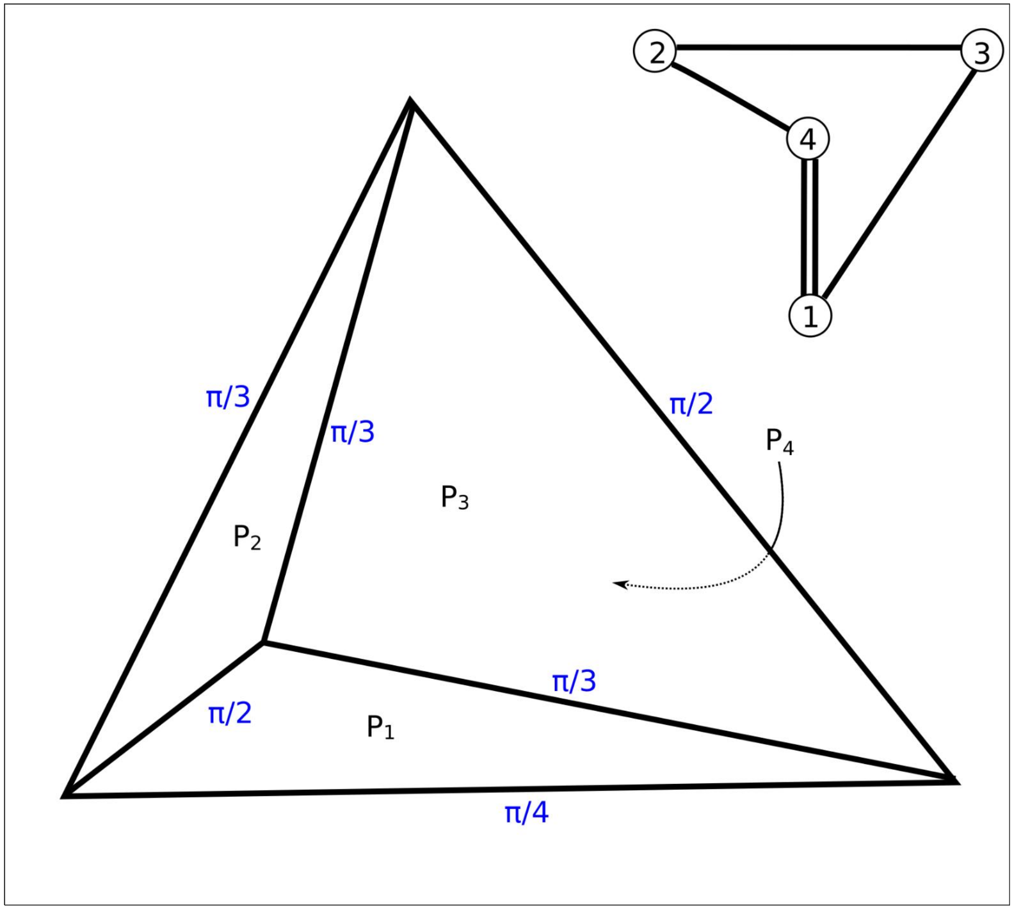


Color count equals to the index of subgroup.

The index of subgroup of infinite group can be arbitrary large.

This is coset coloring with 10 colors

Building Hyperbolic Tiling



Simplest hyperbolic tilings are tiling with tetrahedra.

Lets start with specific example.

Hyperbolic tetrahedron with all finite vertices.

It is convenient to display tetrahedra using Coxeter diagram.

Each plane is shown as small circle.

For each pair of faces (i, j) with dihedral angle $\alpha = \frac{\pi}{m_{ij}}$ the corresponding pair of circles i, j is connected with $(m_{ij} - 2)$ lines.

Building Tetrahedron 1

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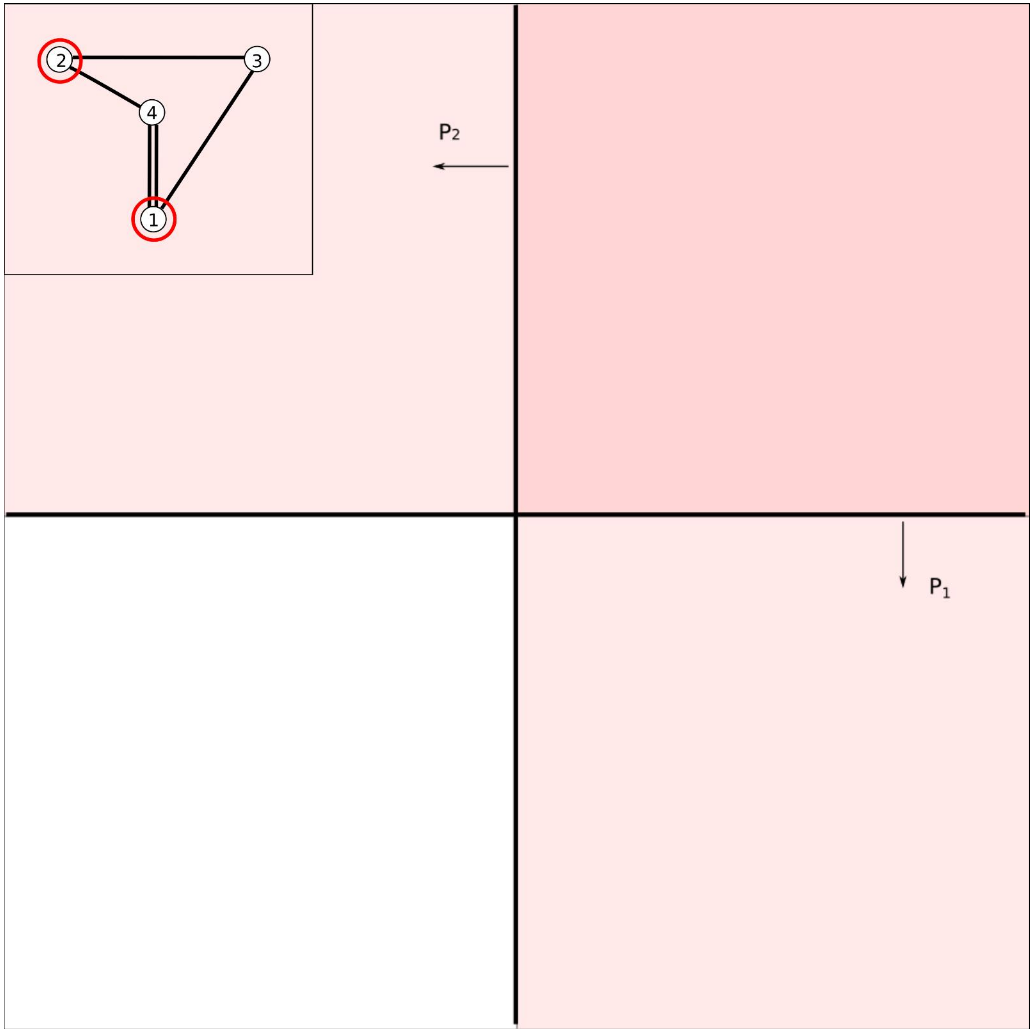
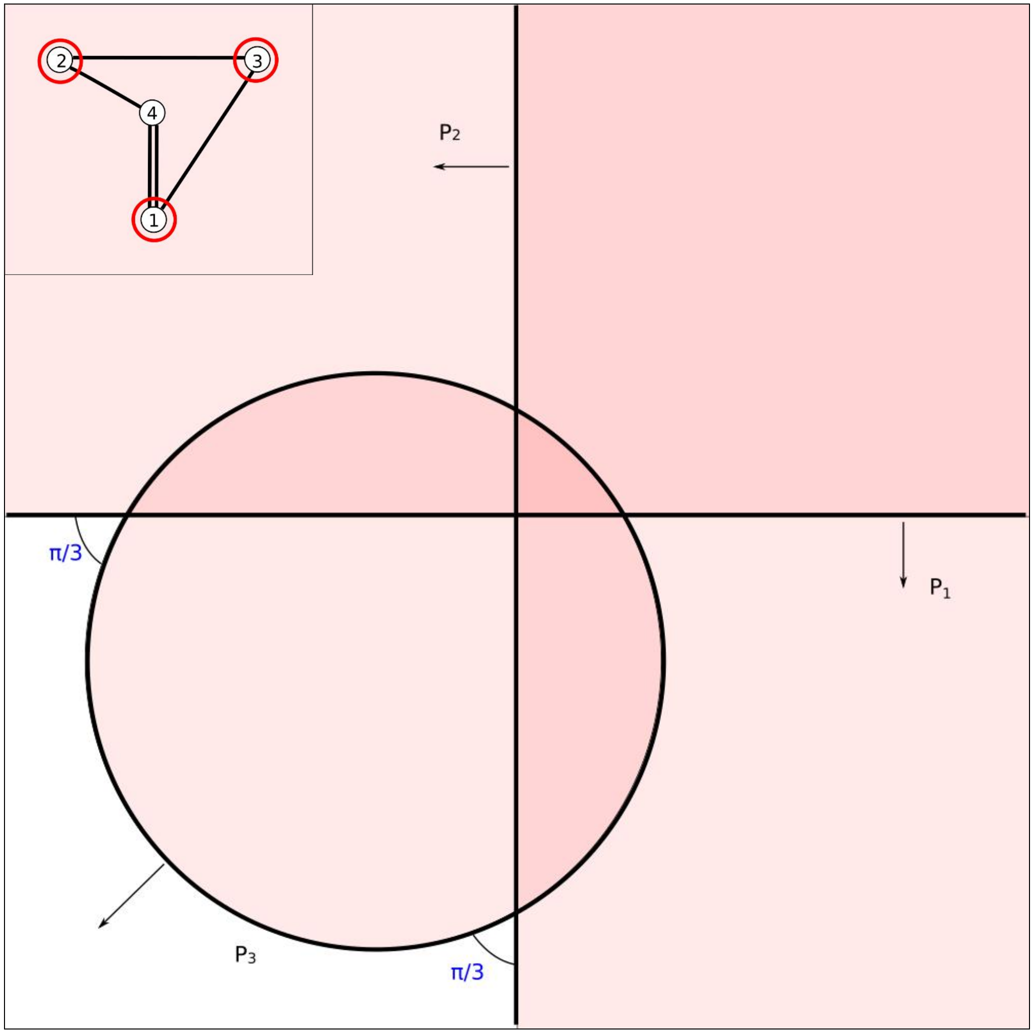


Let's work in the upper half space model of H^3 . Horizon of H^3 lies in the xy-plane $(z = 0)$

Lets place planes P_1 and P_2 orthogonal to Z axis and orthogonal to each other.

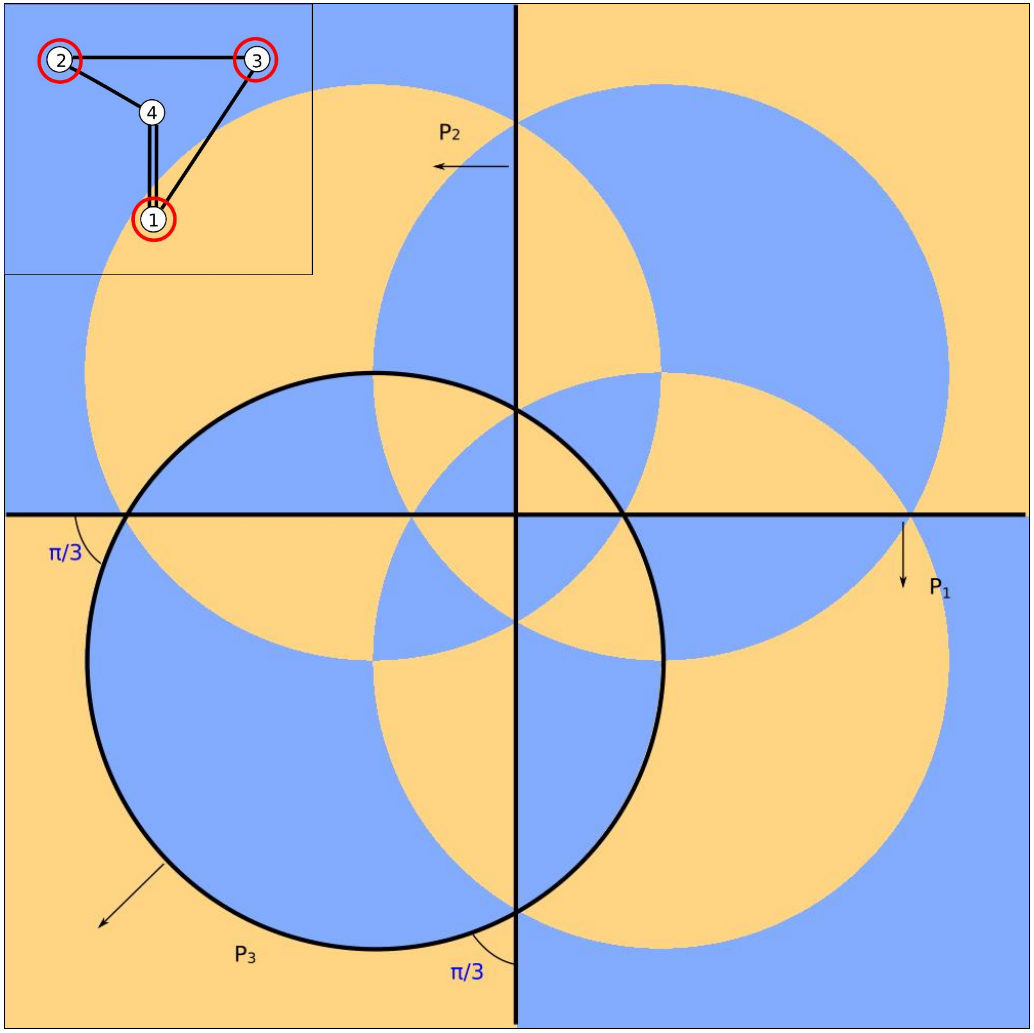
Building Tetrahedron 2



Plane P_3 is a sphere centered at the horizon.

Dihedral angles of its intersections with planes P_1 and P_2 are $\alpha_{13} = \frac{\pi}{3}$ and $\alpha_{13} = \frac{\pi}{3}$.

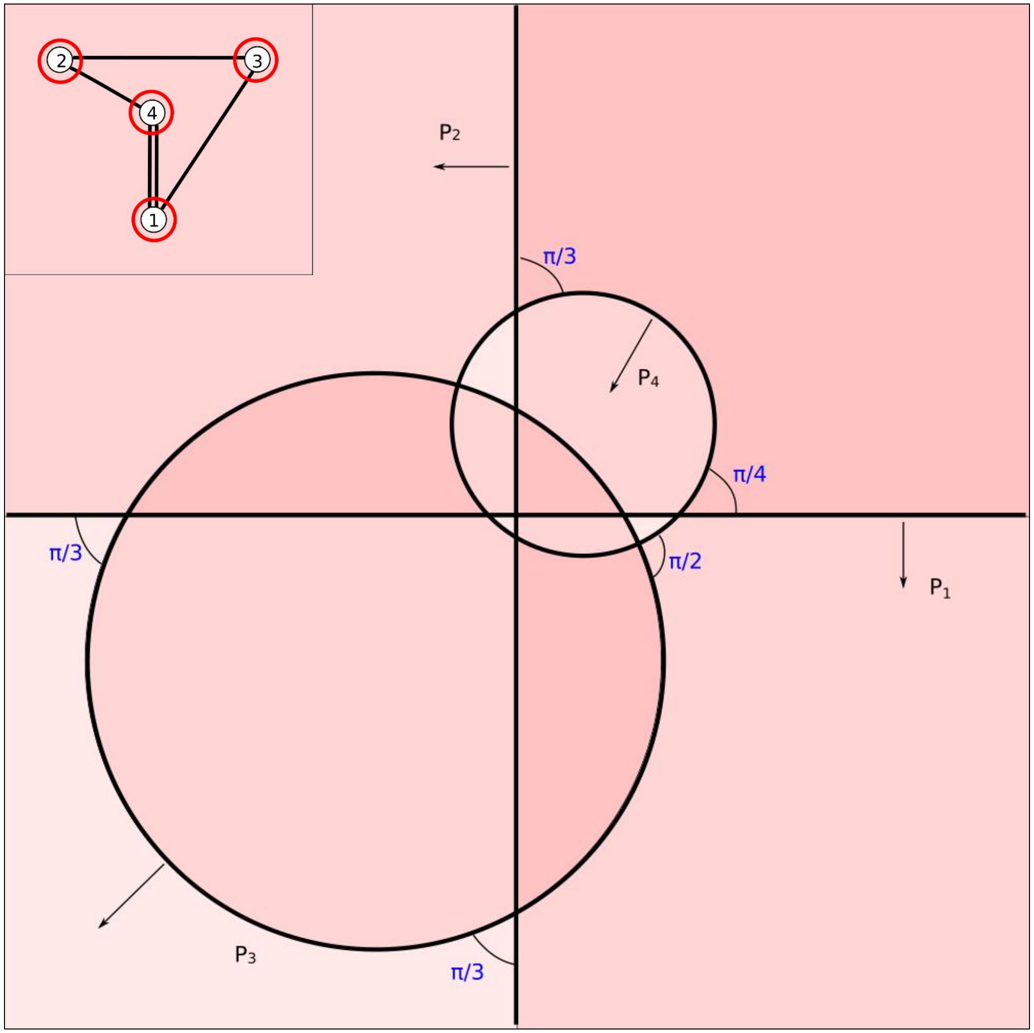
Building Tetrahedron 3



Tiling generated by splanes $P_1\ P_2\ P_3$.

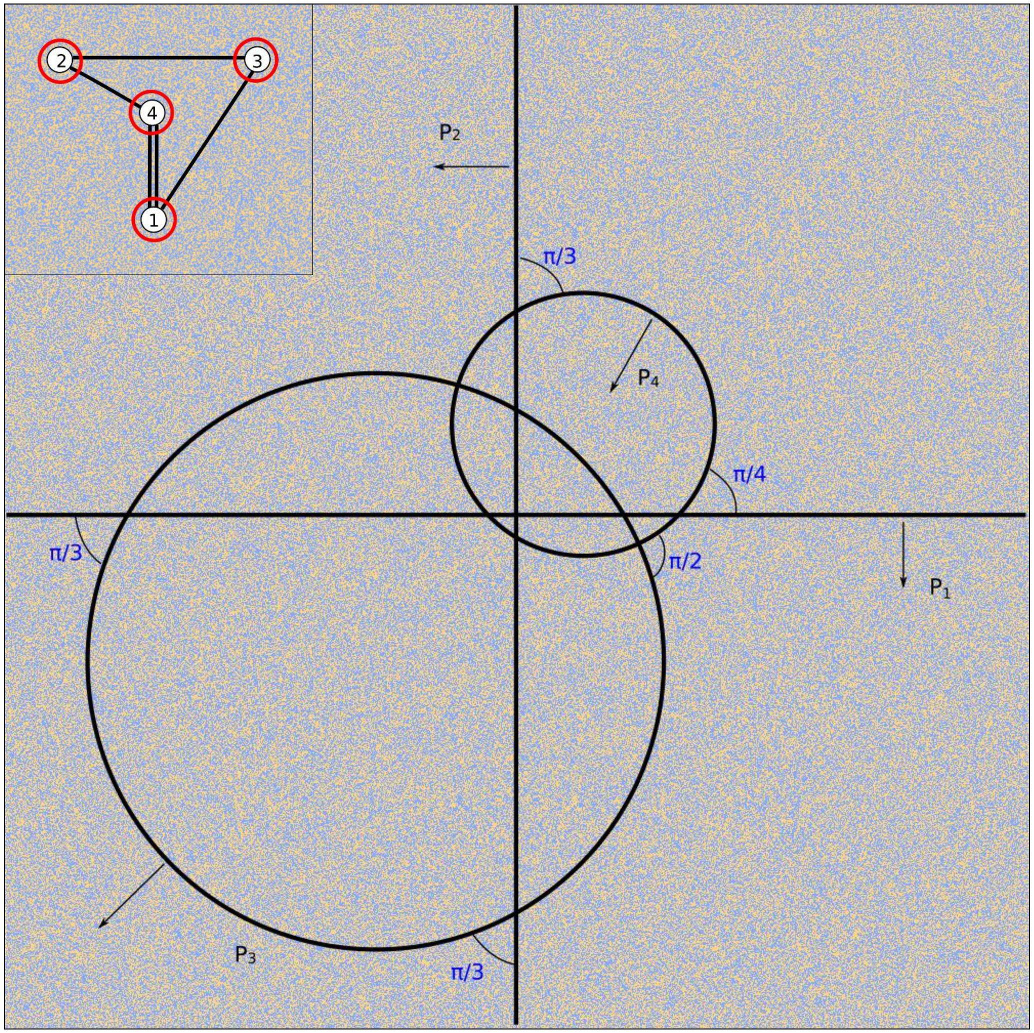
It is stereographic projection of the spherical tiling $\star 332$.

Building Tetrahedron 4



Fourth sphere P_4 with dihedral angles $\alpha_{41} = \frac{\pi}{4}$ $\alpha_{42} = \frac{\pi}{3}$, $\alpha_{43} = \frac{\pi}{2}$ finishes the hyperbolic tetrahedron.

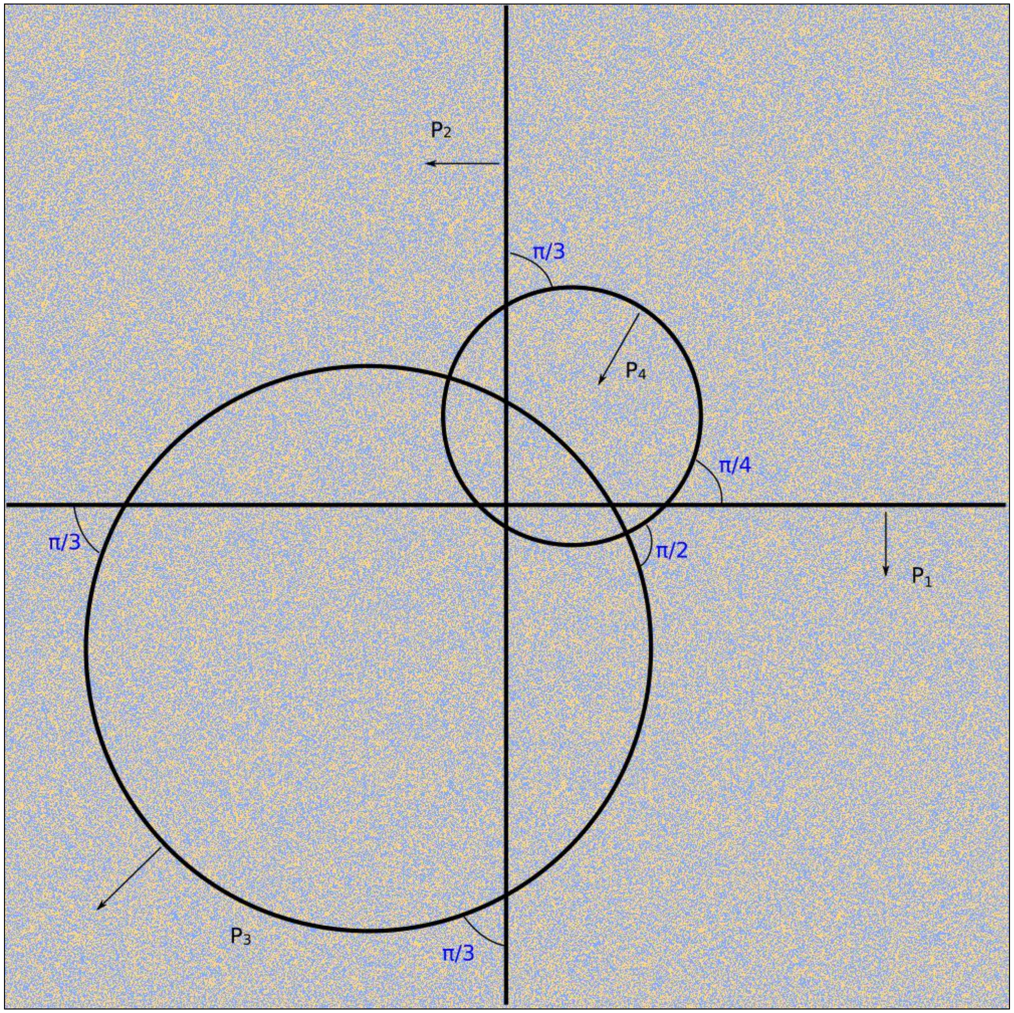
Tiling by tetrahedra



Here is the tiling.

What is going on?

Tiling by tetrahedra



This is the tiling.

What is going on?

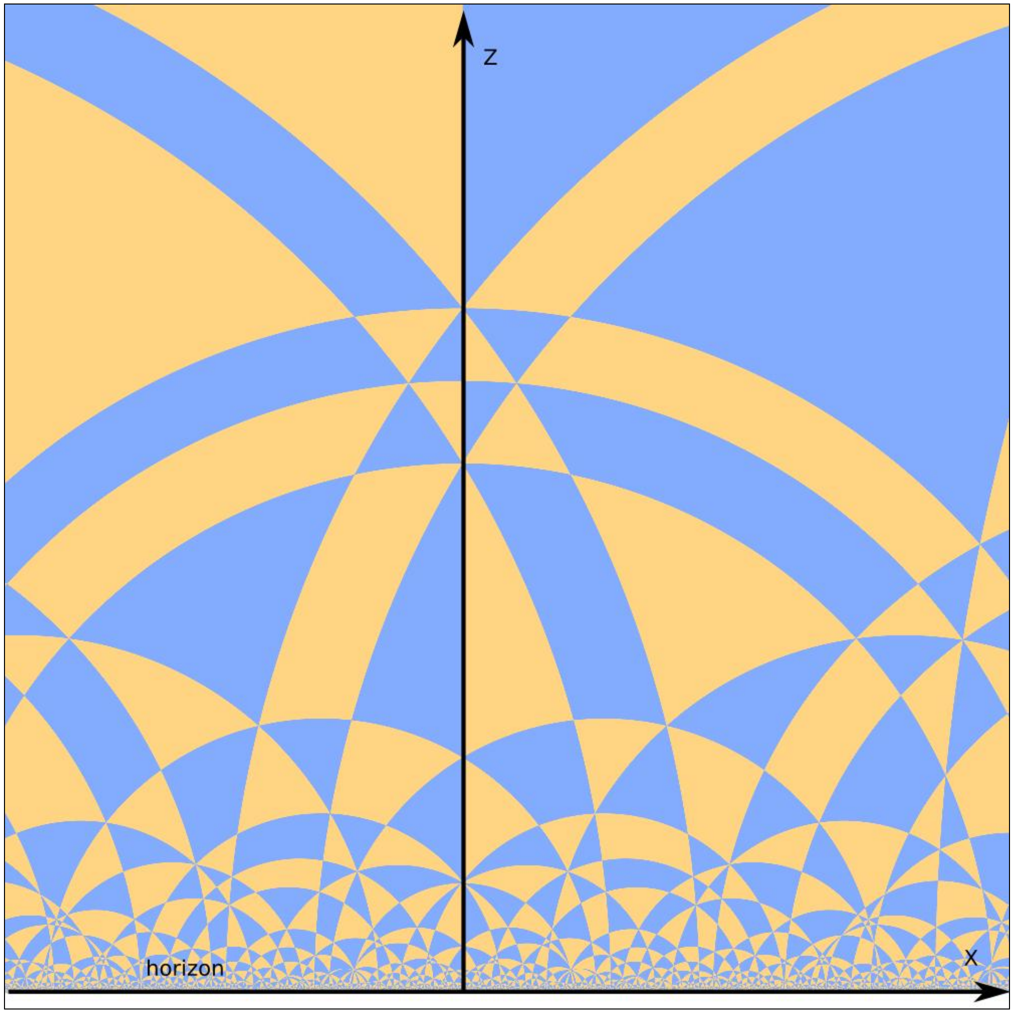
We are looking in the wrong place!

Plane $z = 0$ is the horizon (infinity) of H^3.

Tiles become infinitely small at the horizon.

The visible noise is due to finite precision of calculation.

Tiling in hyperbolic planes



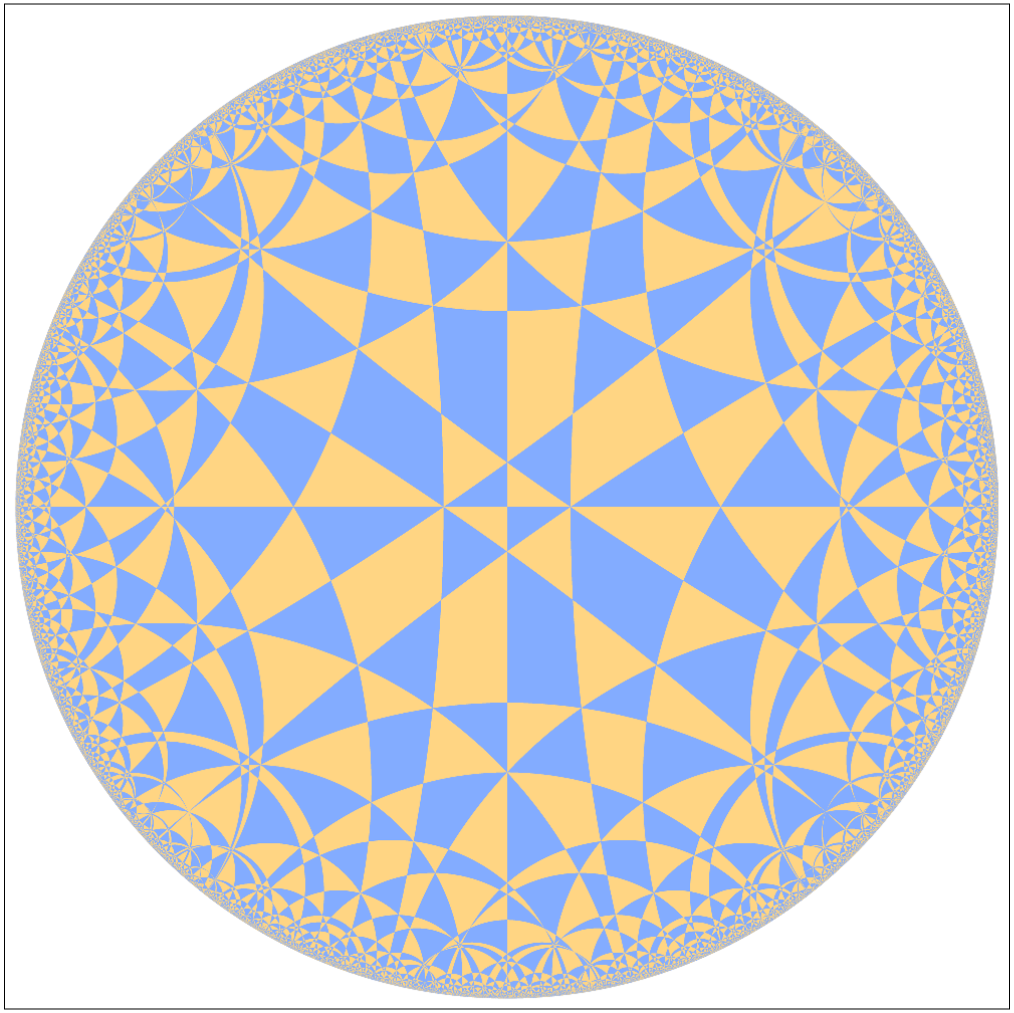
"Right" places to look at the space tiling are hyperbolic planes.

One kind of hyperbolic planes in the upper half space model are planes orthogonal to the horizon.

The image looks similar to 2D tiling of hyperbolic plane in the upper half plane model of H^2.

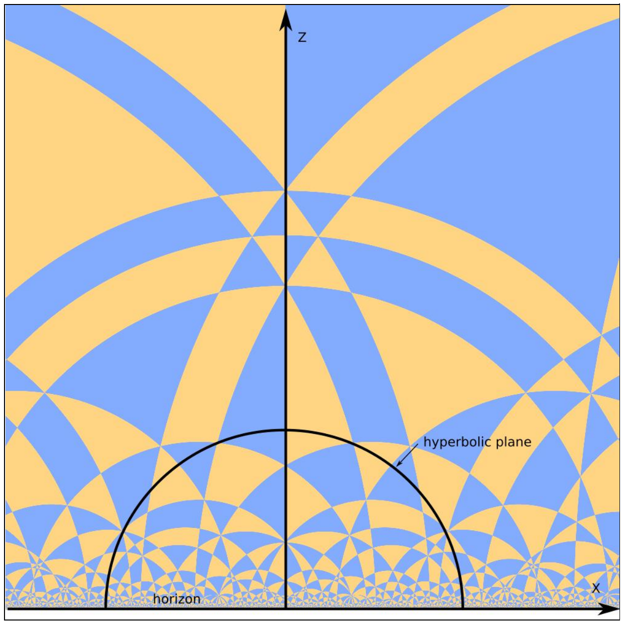
However the polygons are not 2D tiles, but odd shaped slices of 3D tiles by hyperbolic plane.

Tiling in hyperbolic planes



Another type of hyperbolic planes are half spheres centered at the horizon (image below). The planes can be flattened by stereographic projection.

The flattened tiling (on the left) looks similar to the 2D tilings in the Poincare circle model of H^2.

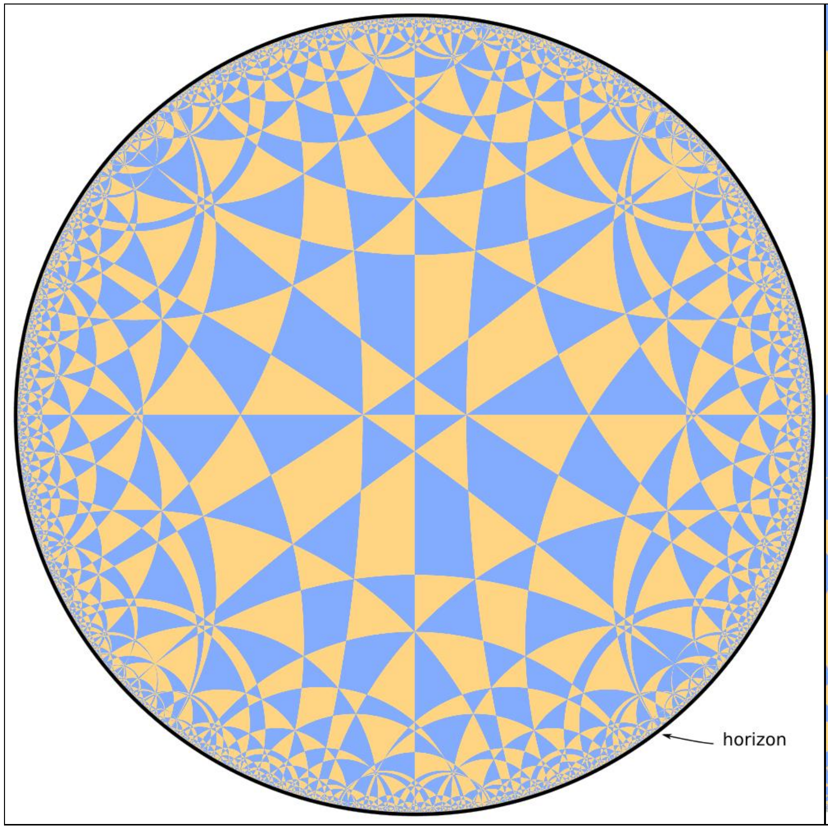


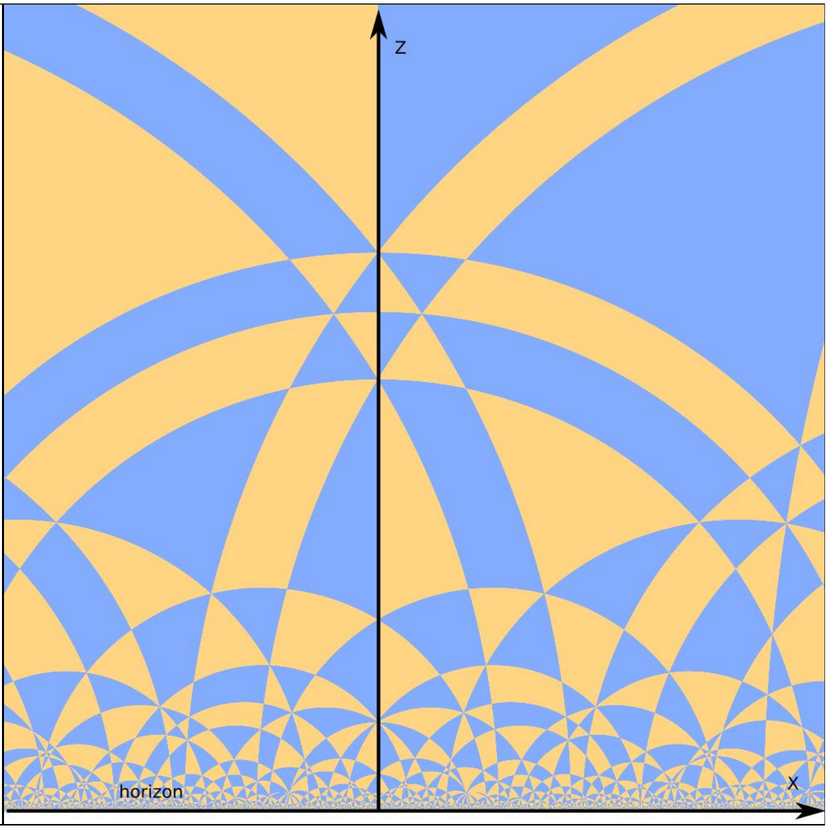
"Problems" of hyperbolic tilings

The main feature of hyperbolic tilings visualization is high range of visual scale.

In the circle model of H^2 we have few regular sized tiles near the center. Tiles shrink rapidly as we approach the horizon.

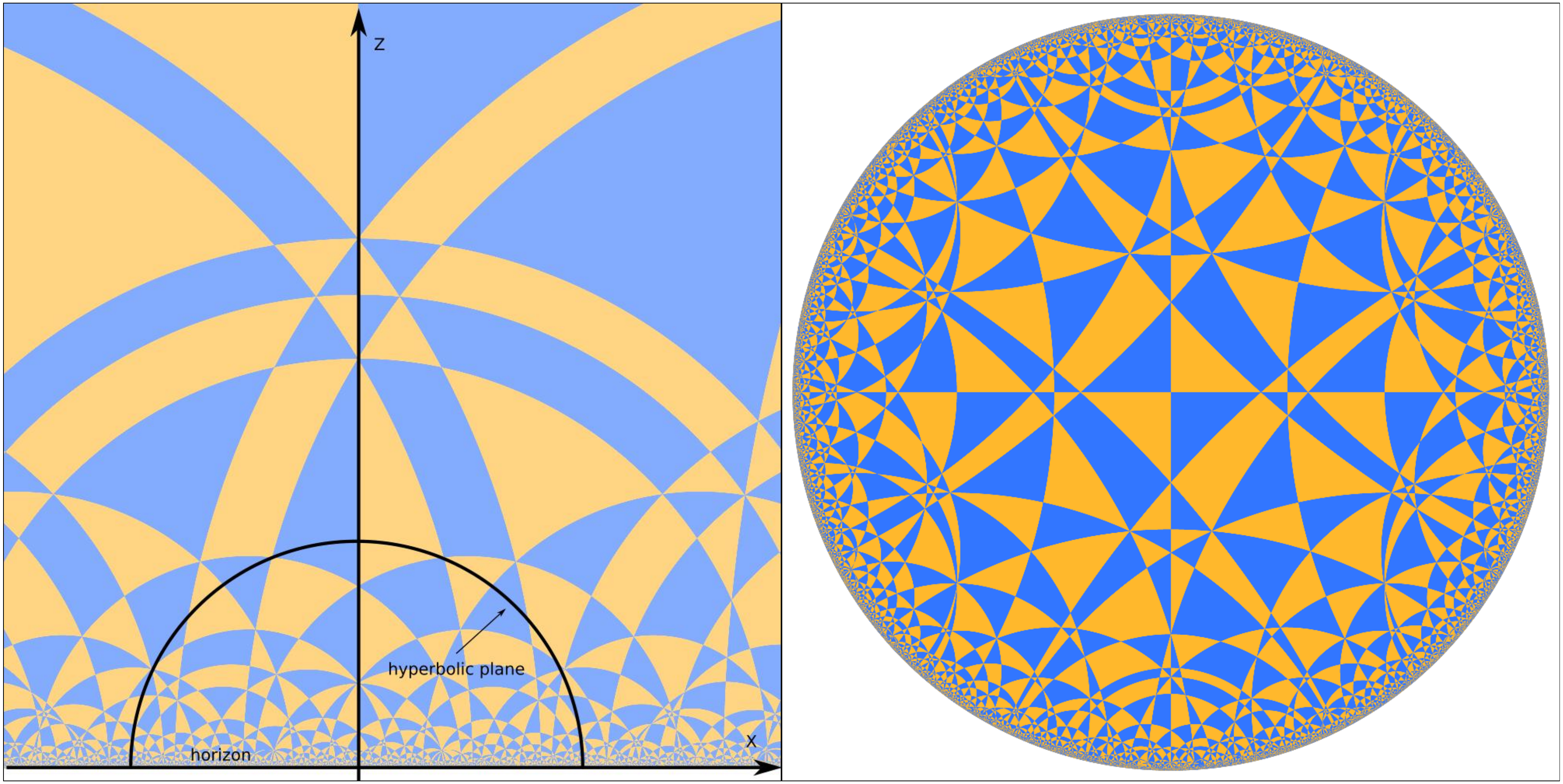
Upper half plane model H^2 has similar behavior. Tiles shrink rapidly as we approach the horizon and stretch rapidly as we move up.





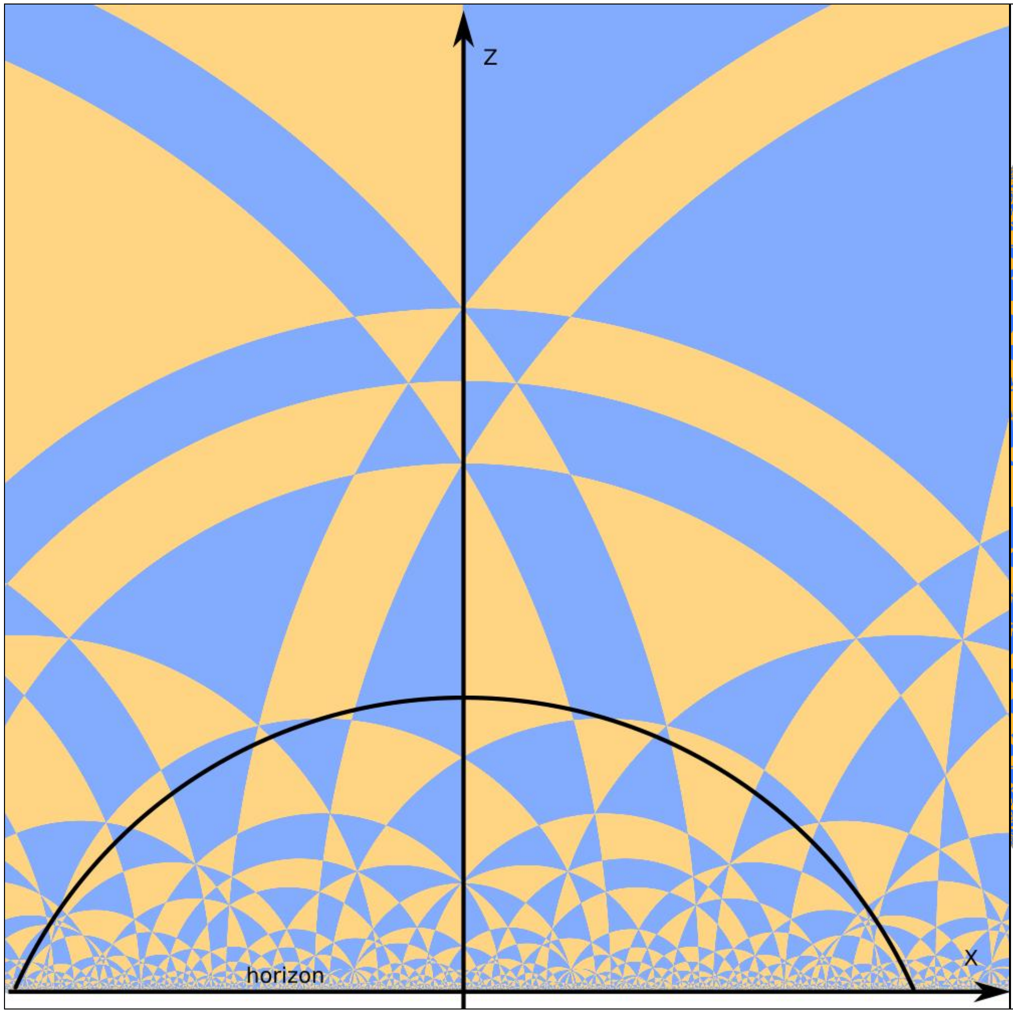
Introducing horospheres

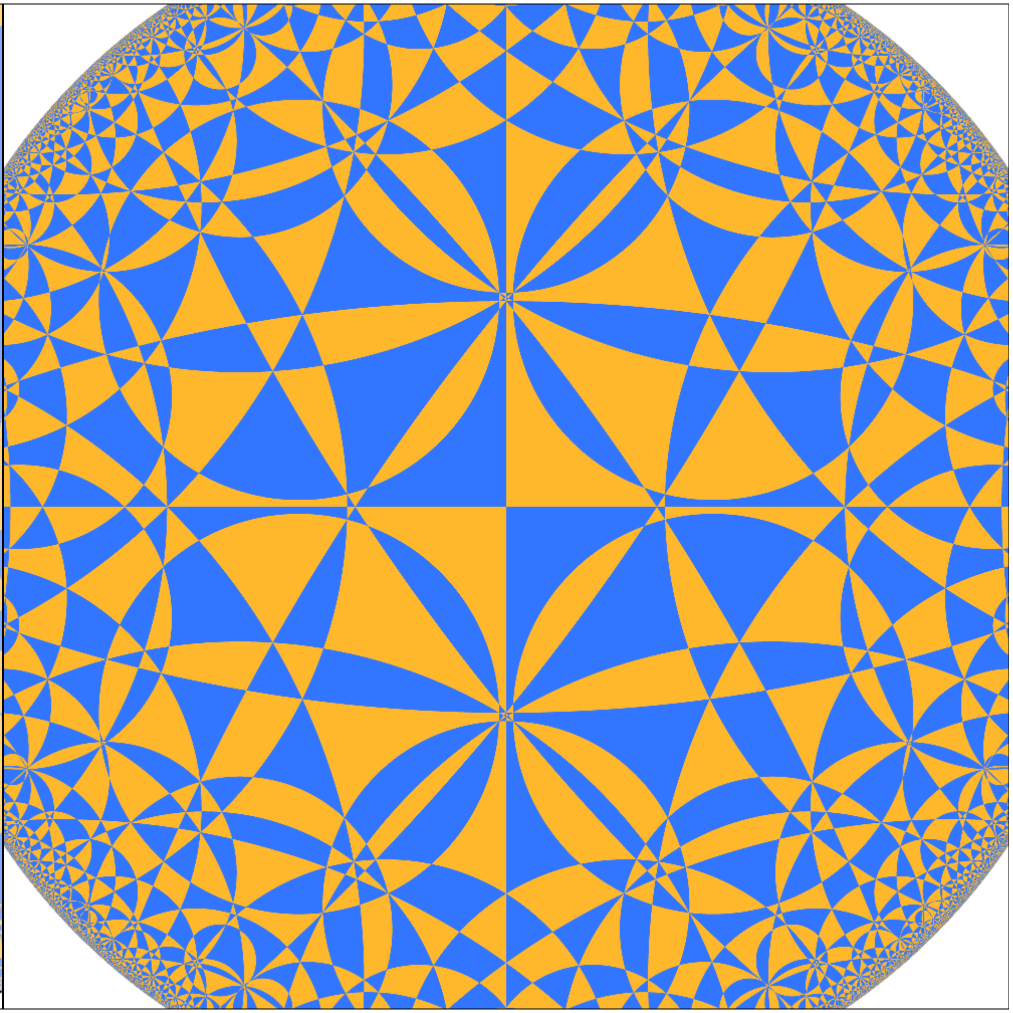
Let's "flatten" the hyperbolic plane.



Horosphere (2)

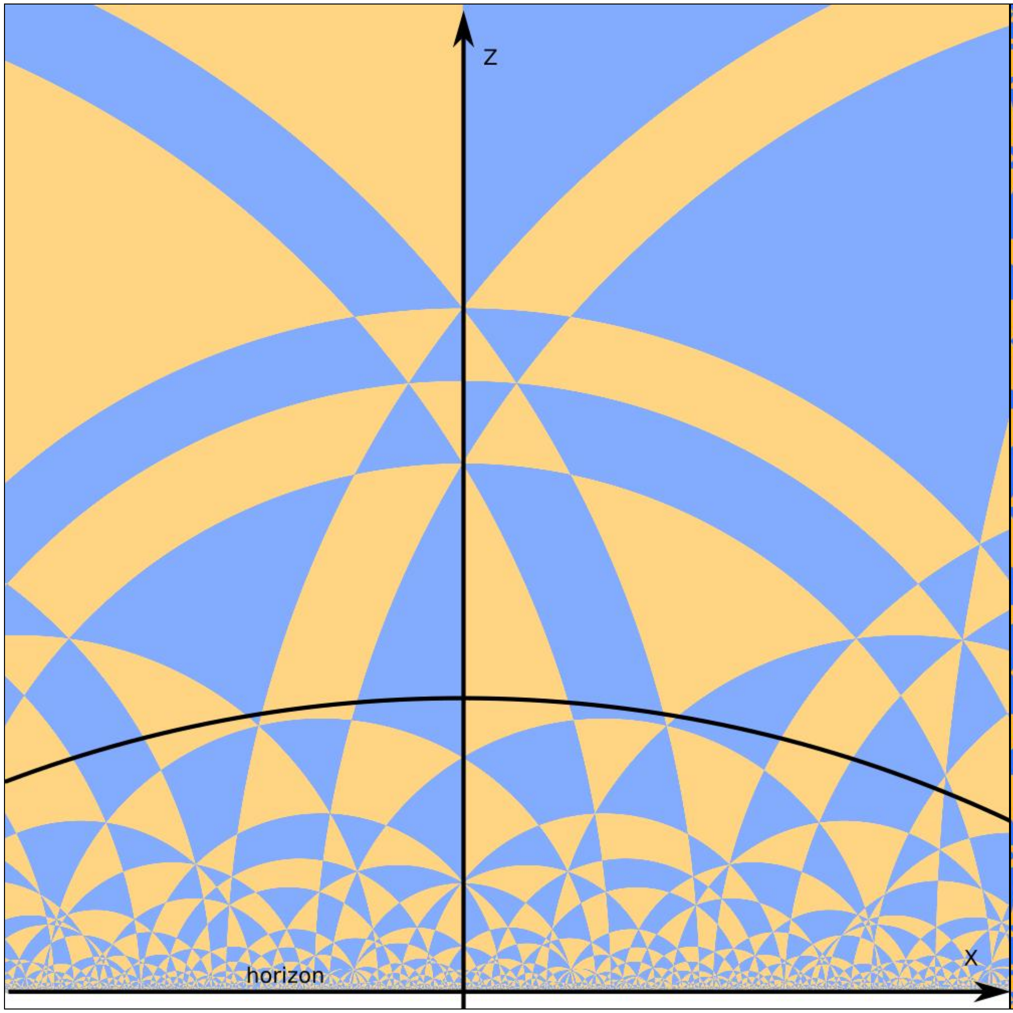
Let's "flatten" the hyperbolic plane.

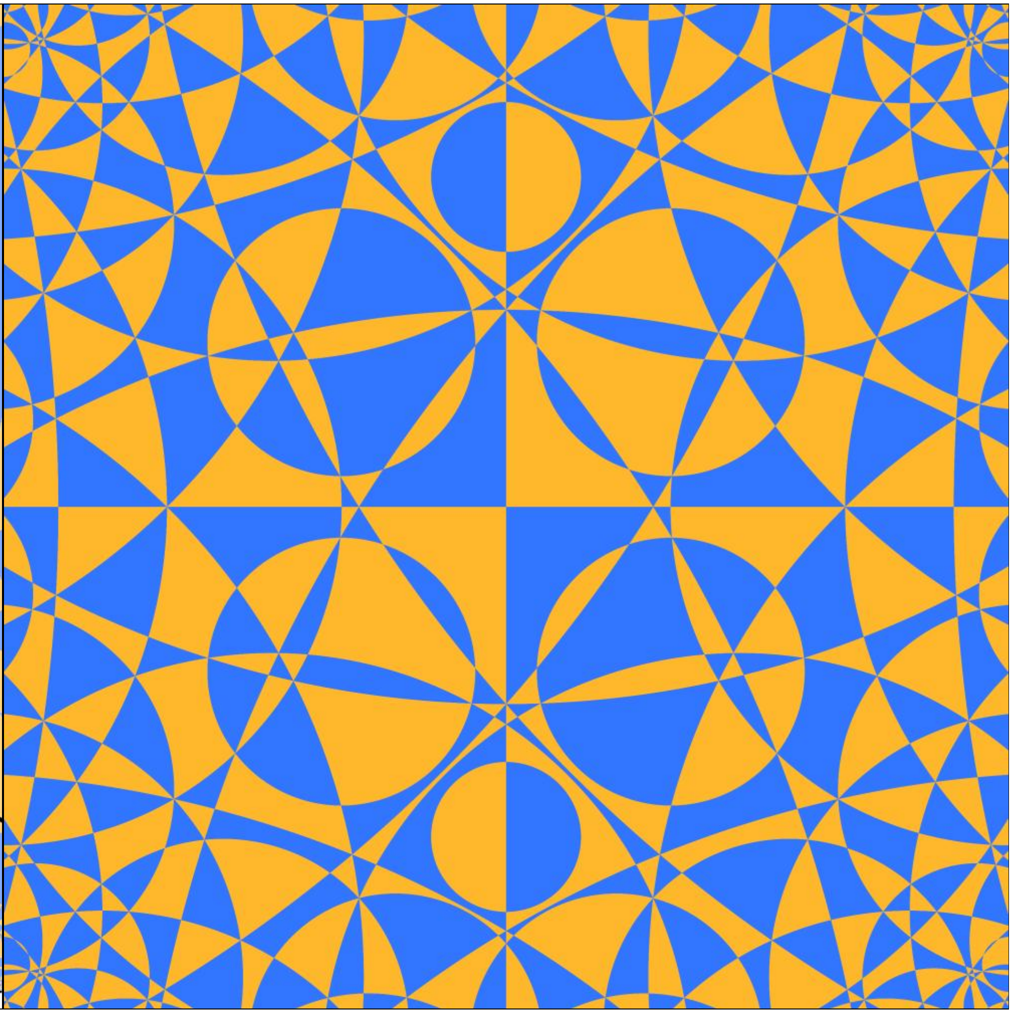




Horosphere (3)

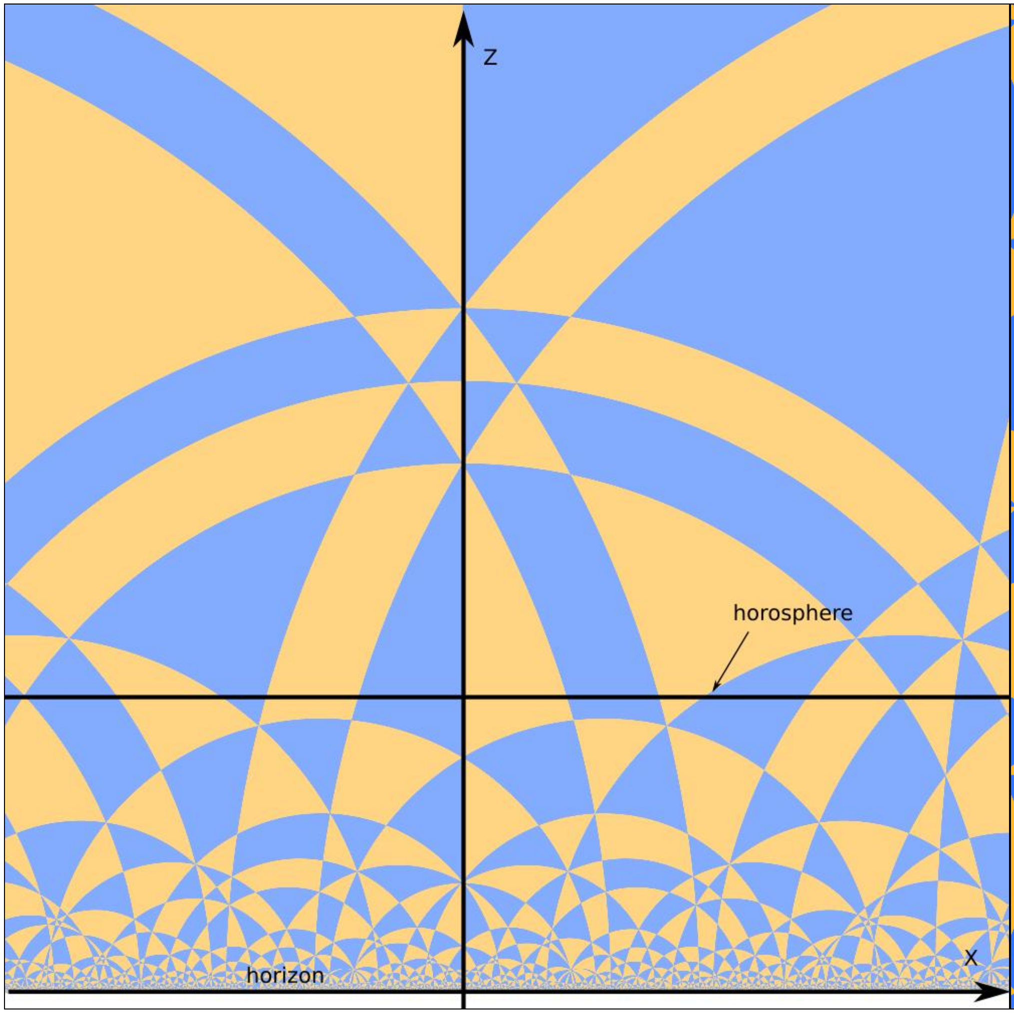
Let's "flatten" the hyperbolic plane.

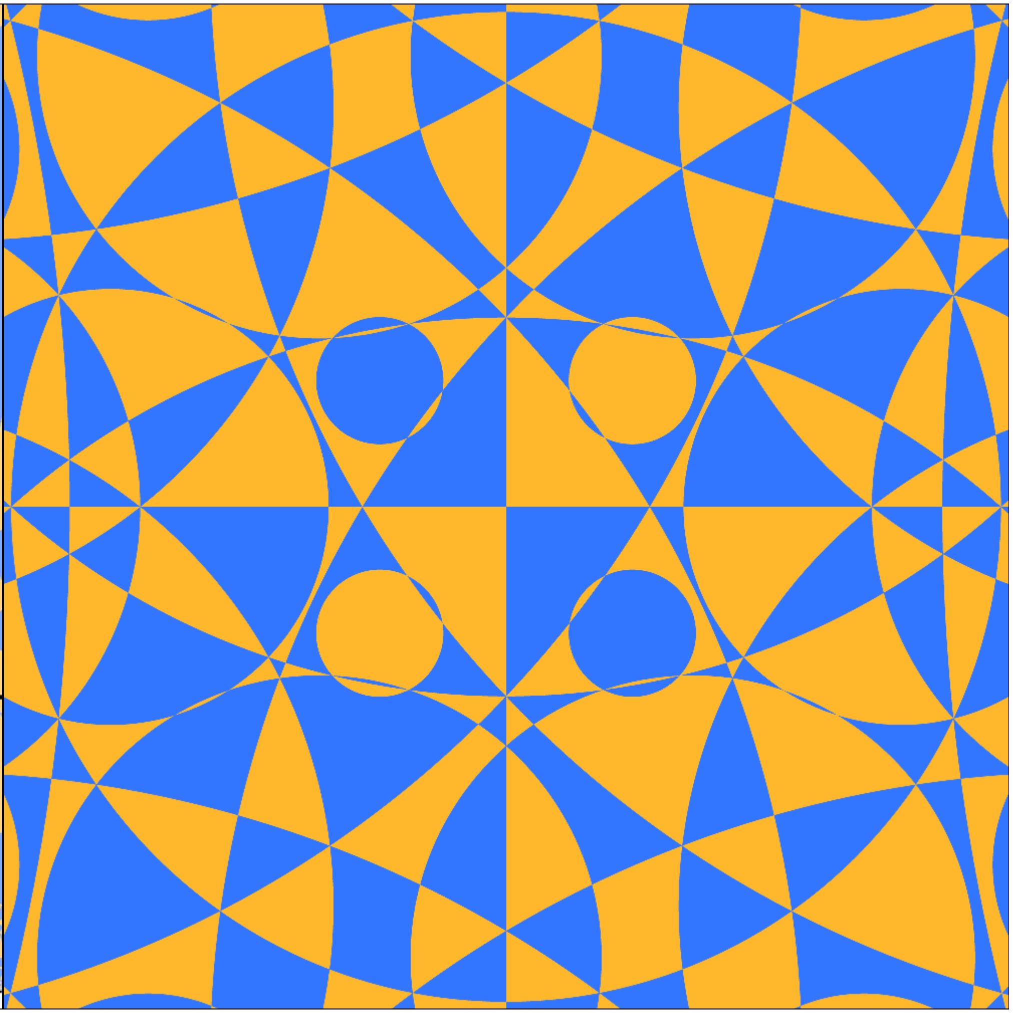




Horosphere (4)

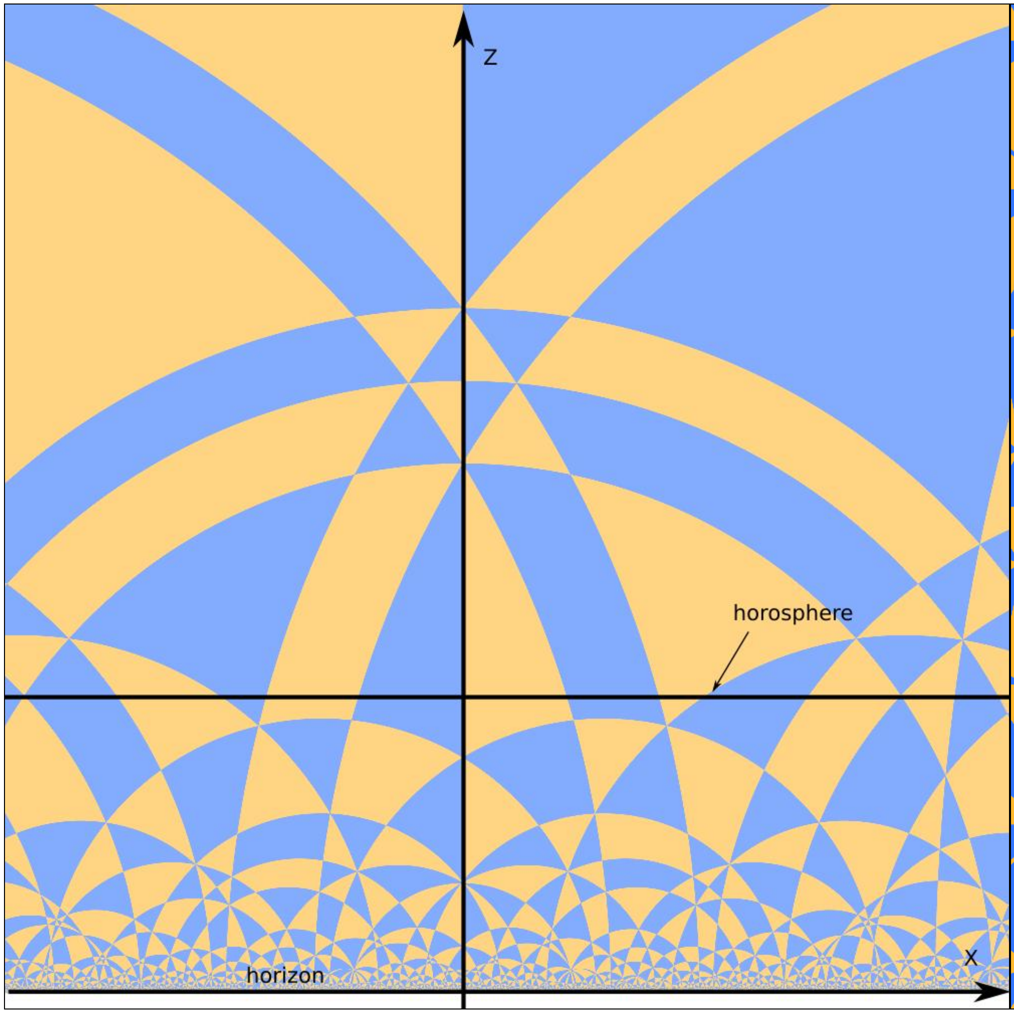
We got horosphere.

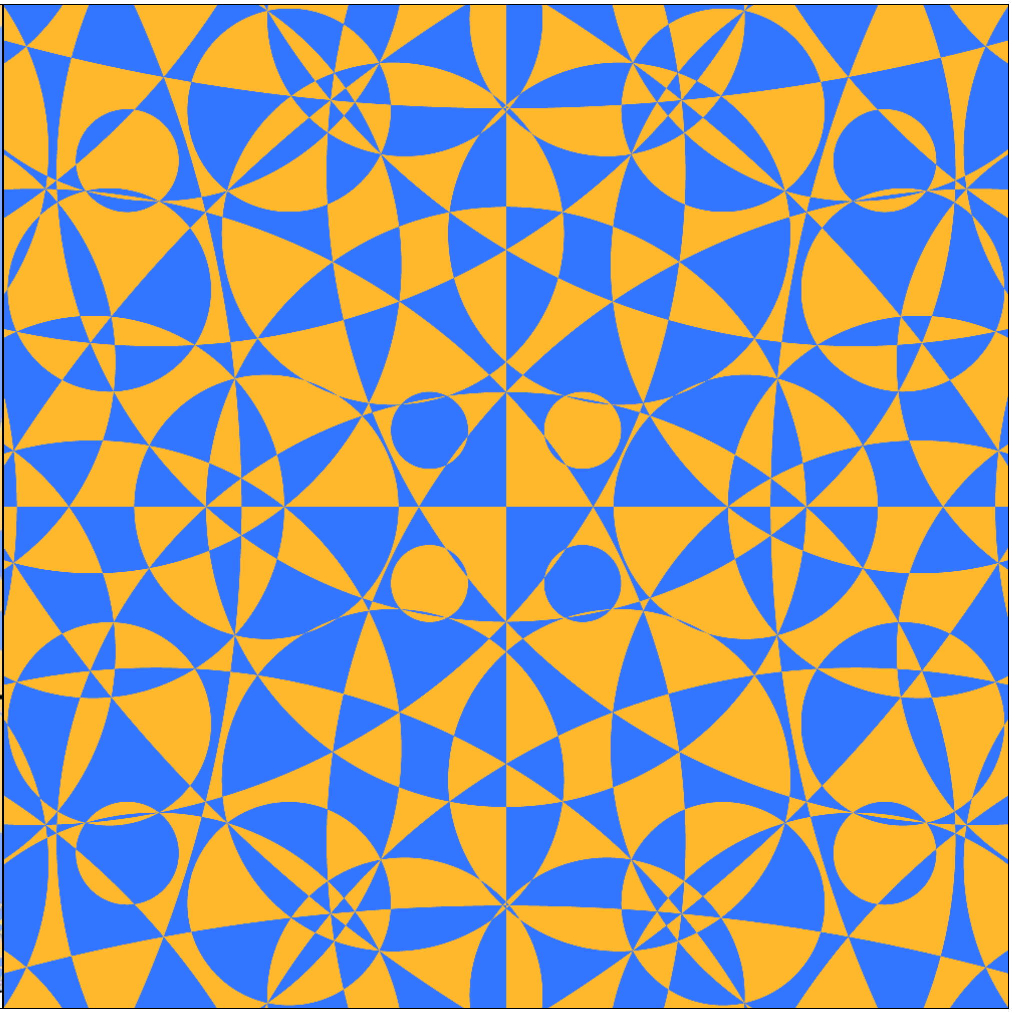




Horosphere (5)

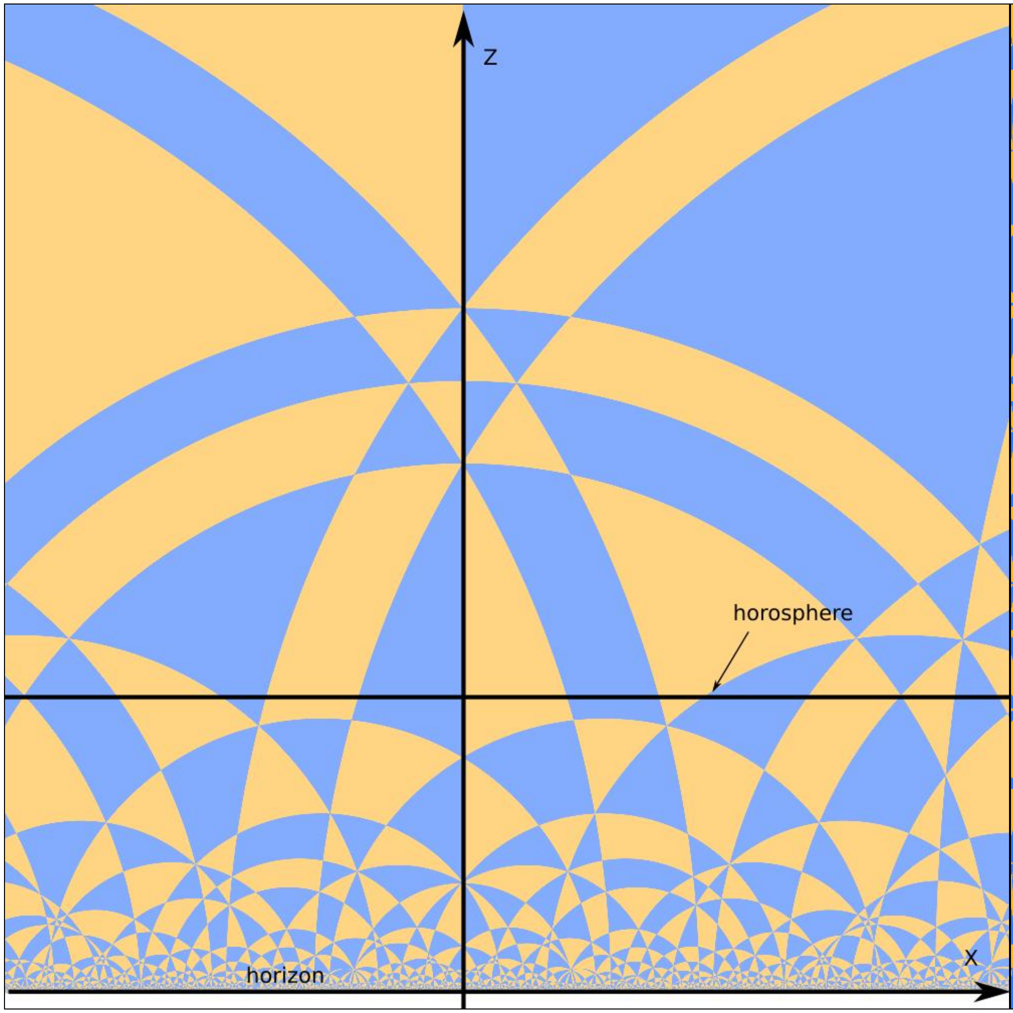
Zooming out the camera.

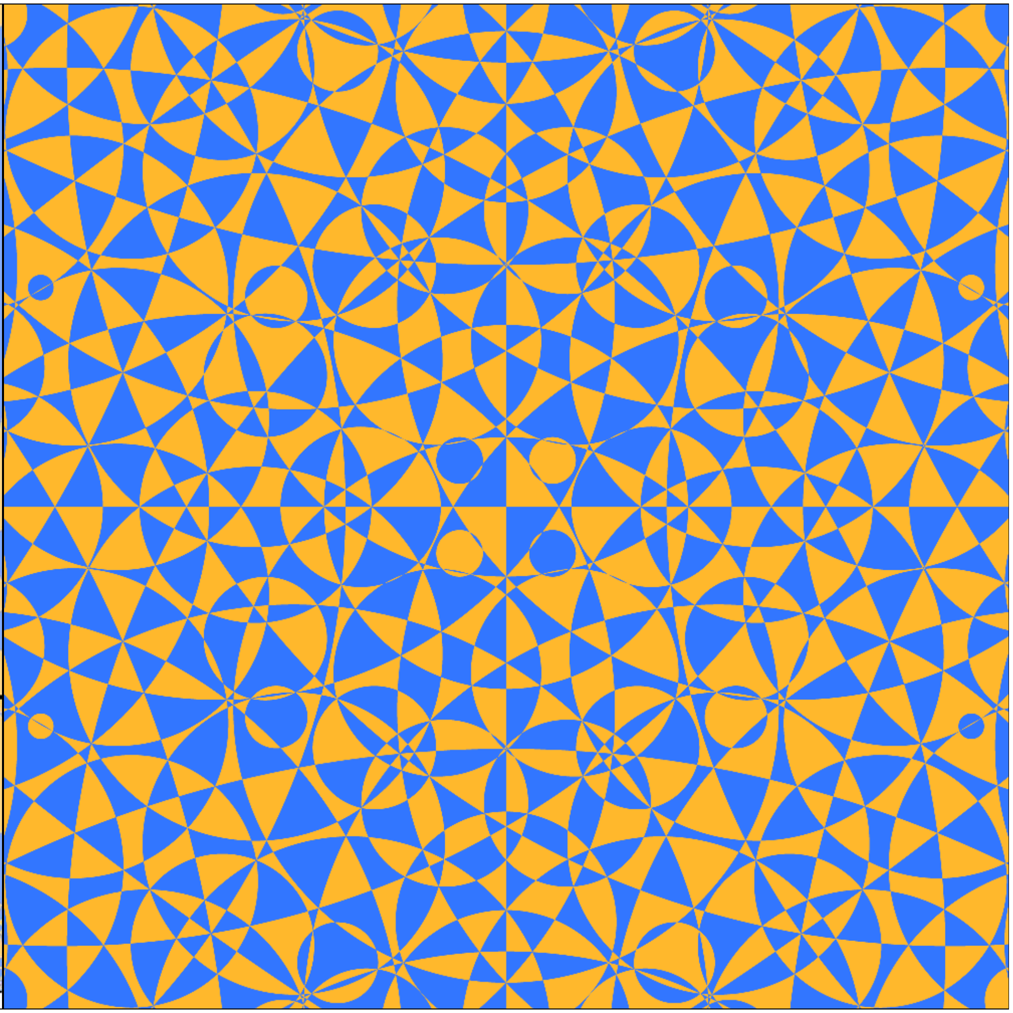




Horosphere (6)

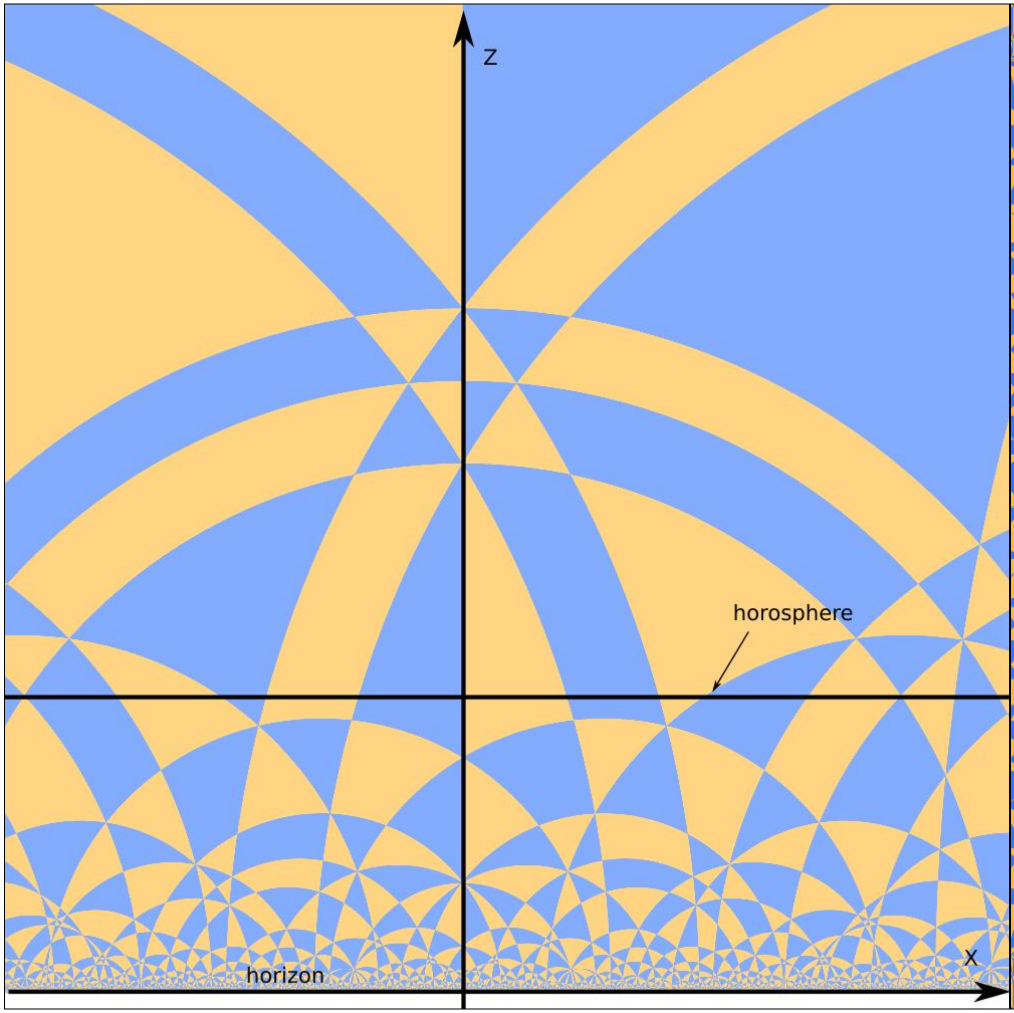
Zooming out the camera.

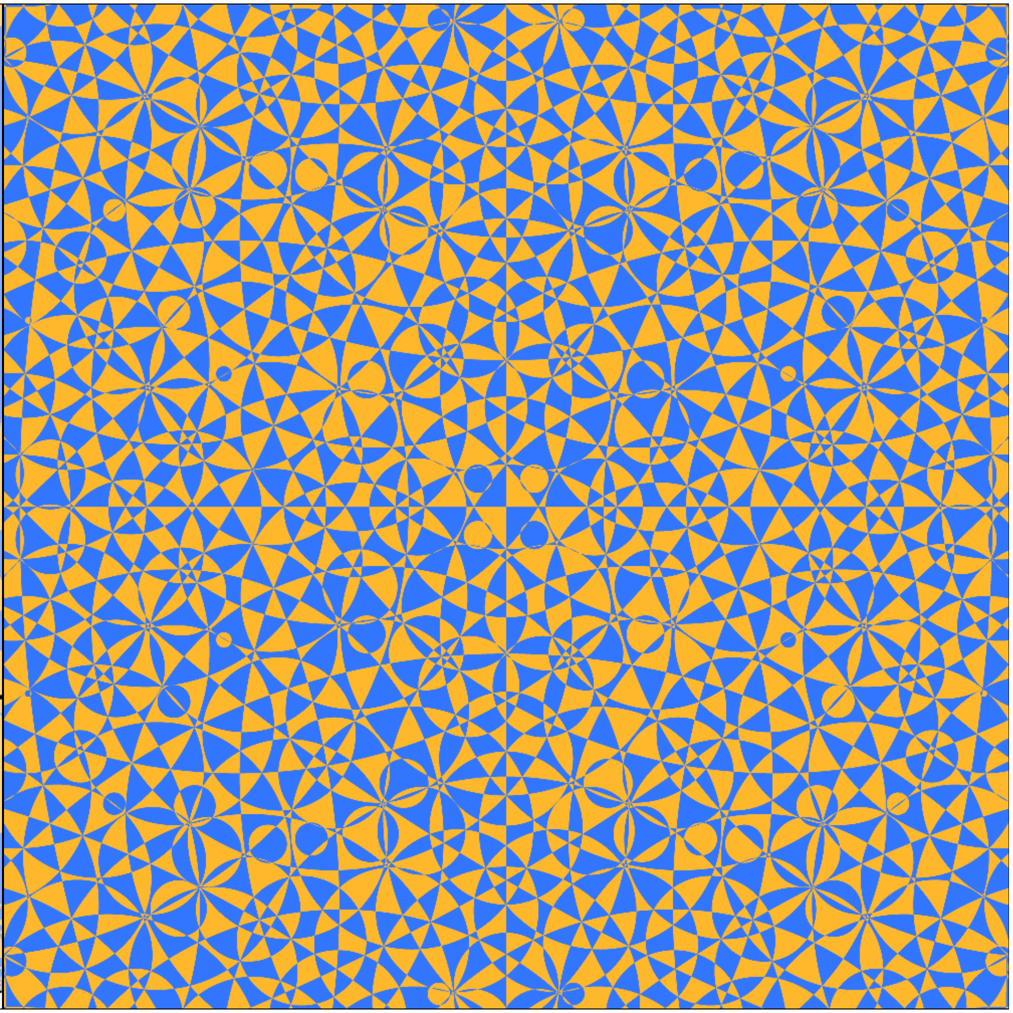




Horosphere (7)

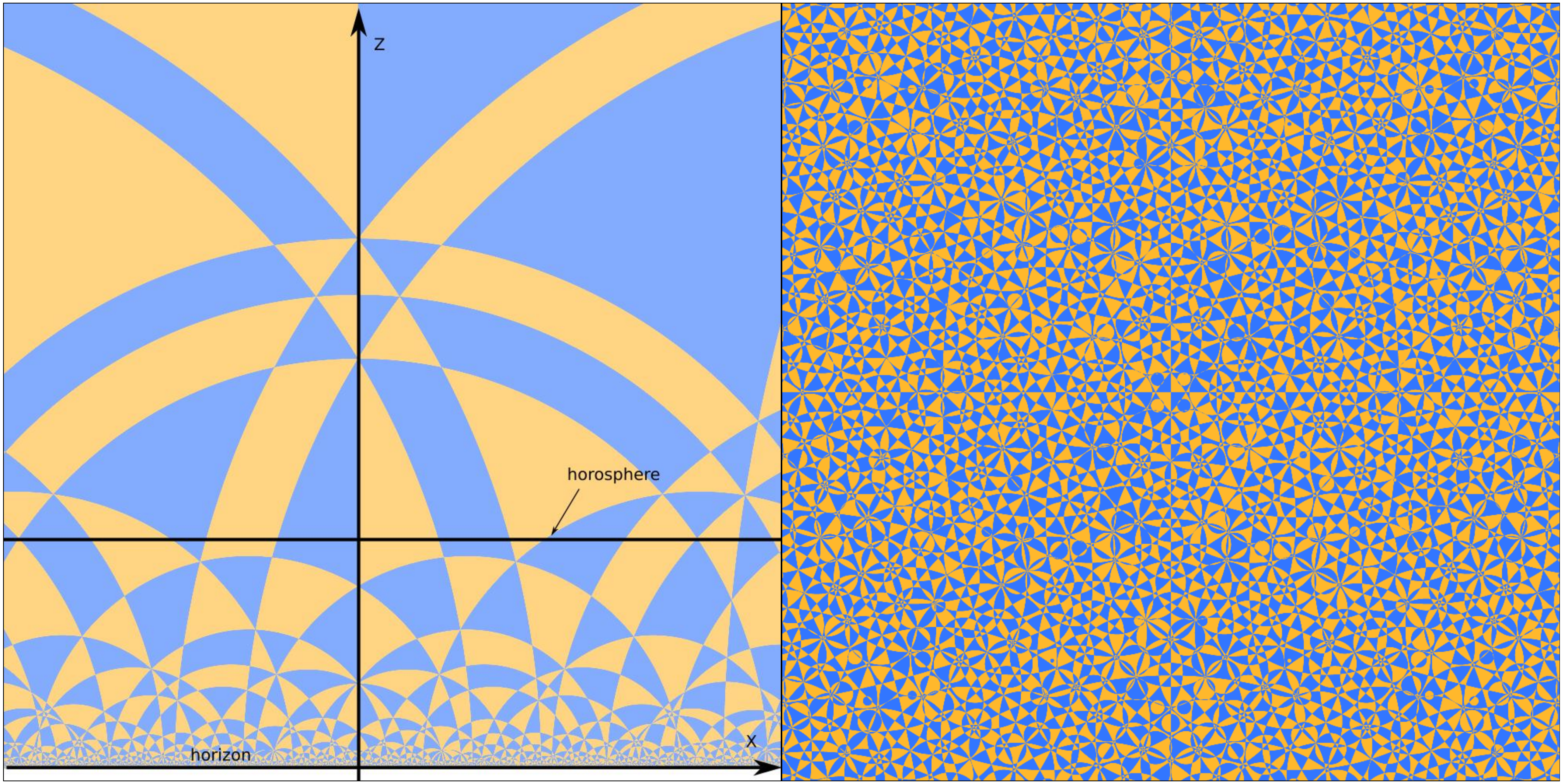
Zooming out the camera.



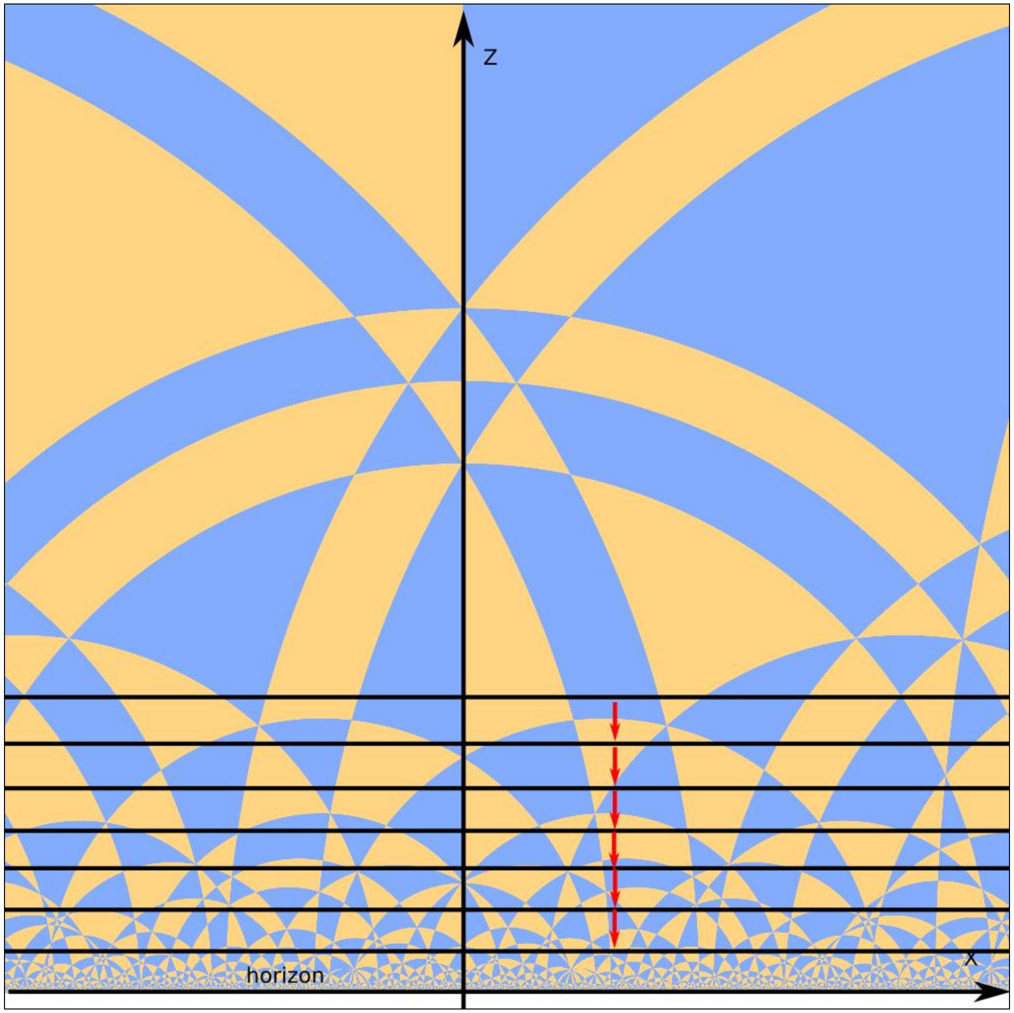


Horosphere (8)

Zooming out the camera. Animaton https://youtu.be/u1Lp9NS98E0



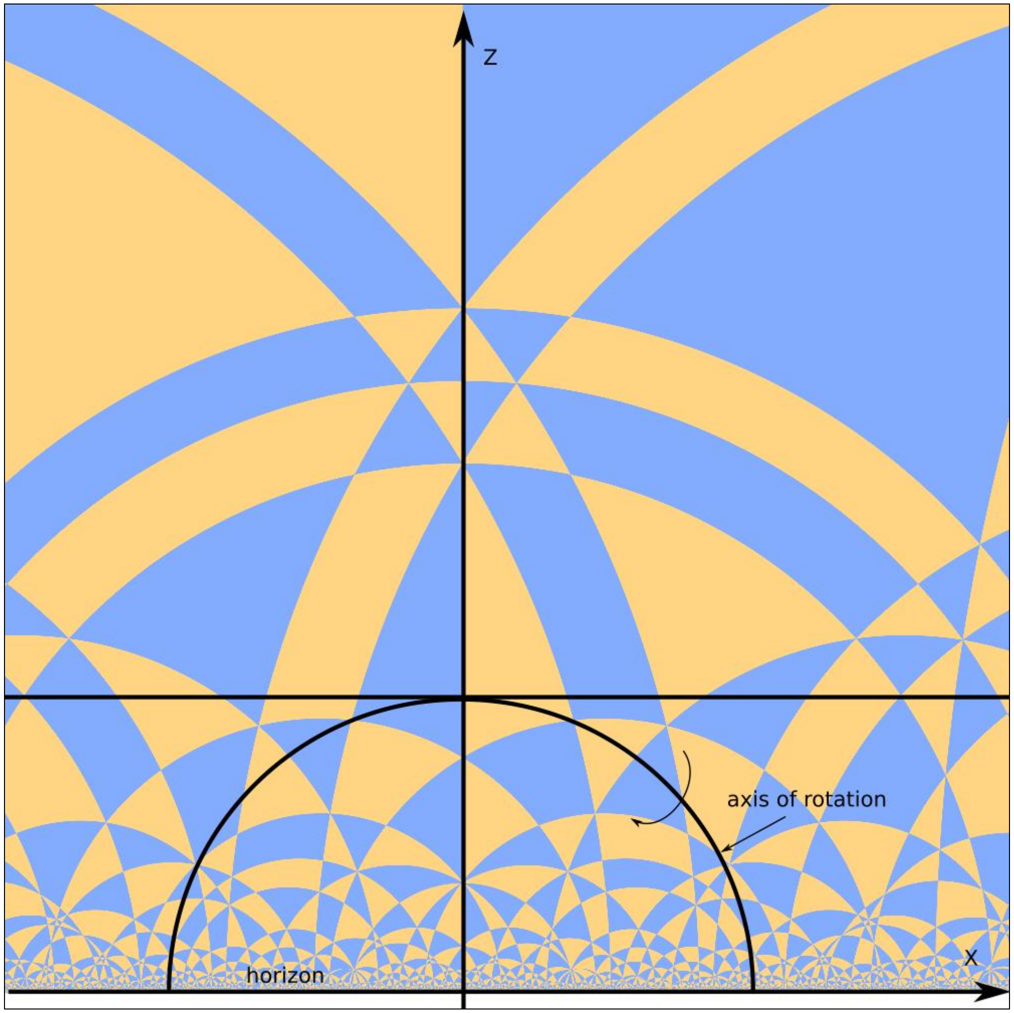
Horosphere Motion



There are different motion we can perform with horospere

Straight hyperbolic translation. Animaton https://youtu.be/PKauqA_Qp9Q

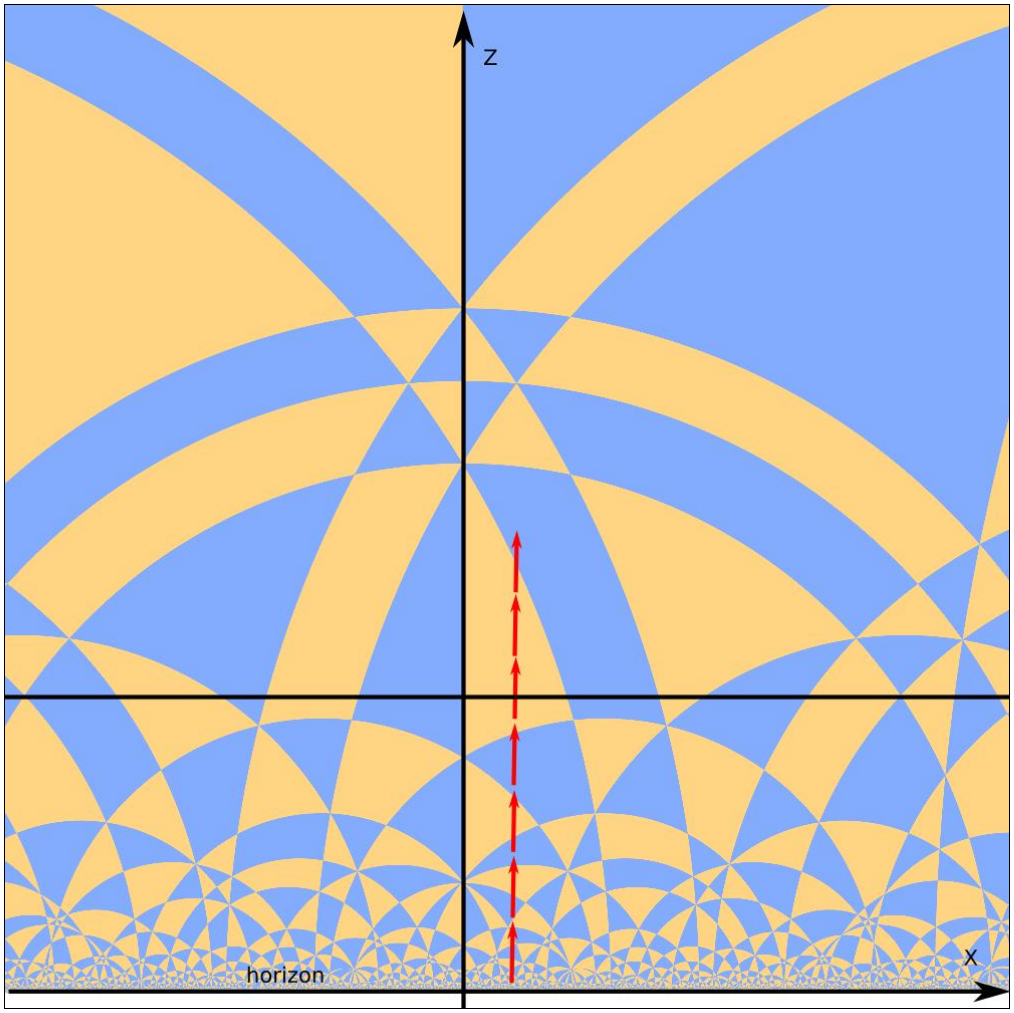
Horosphere Motion 2



There are different motion we can perform with horospere

Hyperbolic rotation around hyperbolic straight line.
Animaton 1.
https://youtu.be/PhcbmsTYc-0
animaton 2.
https://youtu.be/rSEzGtp56vI

Horosphere Motion 3



There are different motion we can perform with horospere

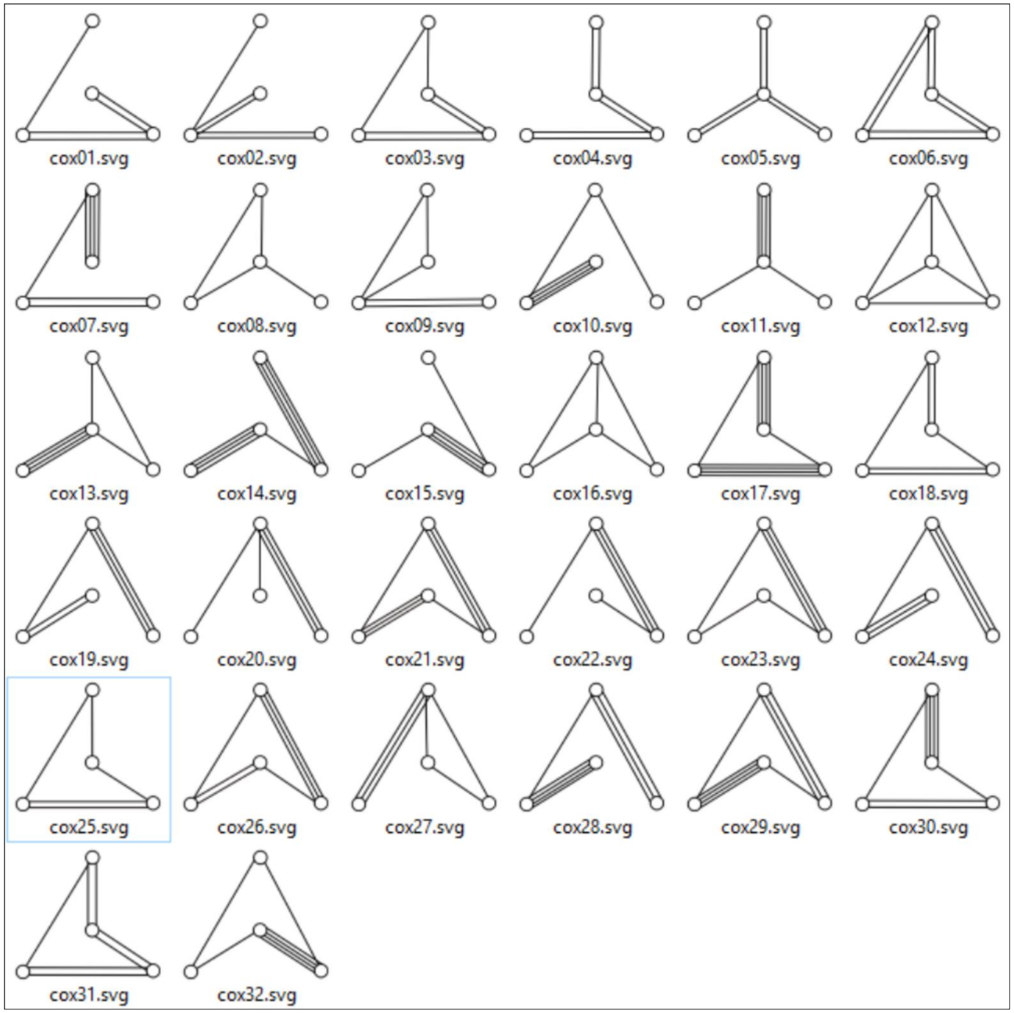
Moving tiling past horosphere.

Similar to horosphere hyperbolic translation.

The difference is that scale of the tiling remain the same.

Animation.
https://youtu.be/gGvlkomzvAg

Different Hyperbolic Tetrahedra



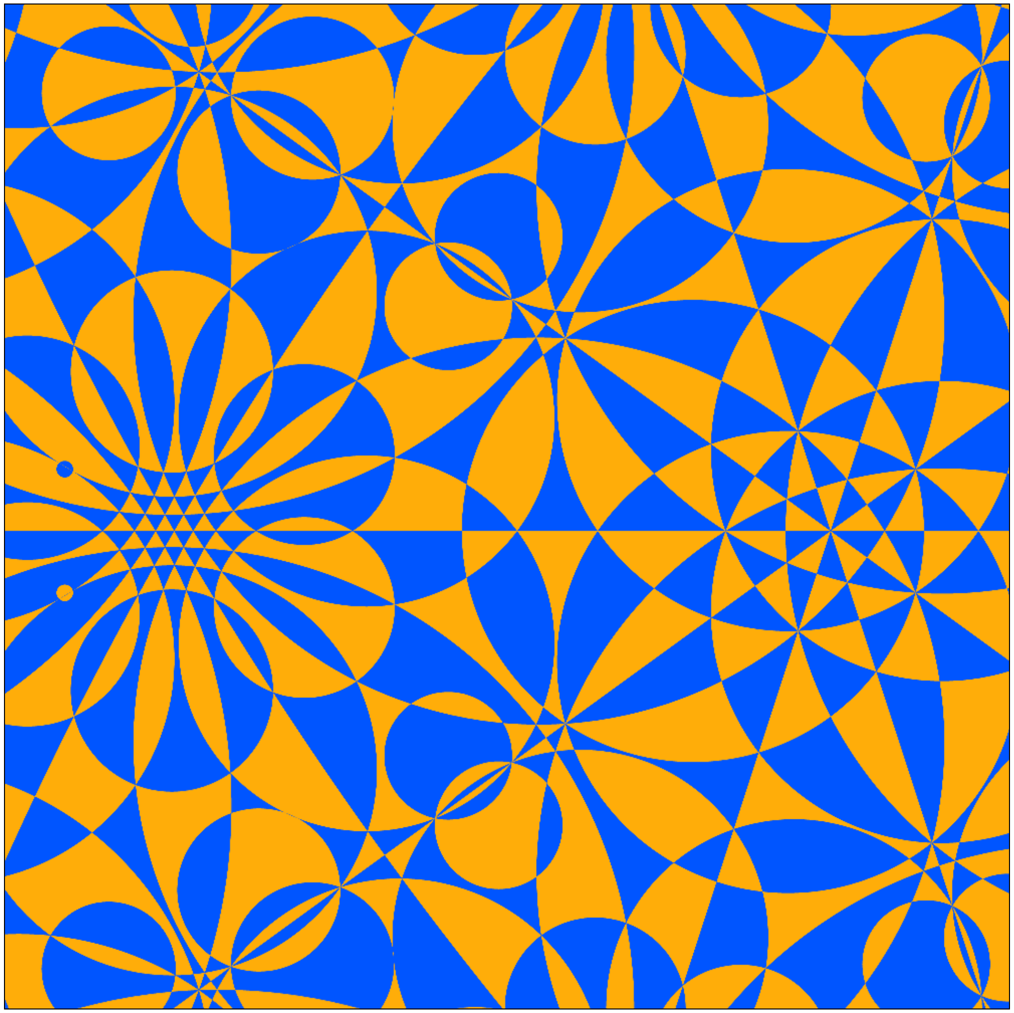
There are 32 compact hyperbolic tetrahedra with kaleidoscopic angles.

There is infinite number of non compact tetrahedra (tetrahedra with vertices beyond infinity).

There is infinite number of more complex polyhedra.

A lot of possibilities for fun.

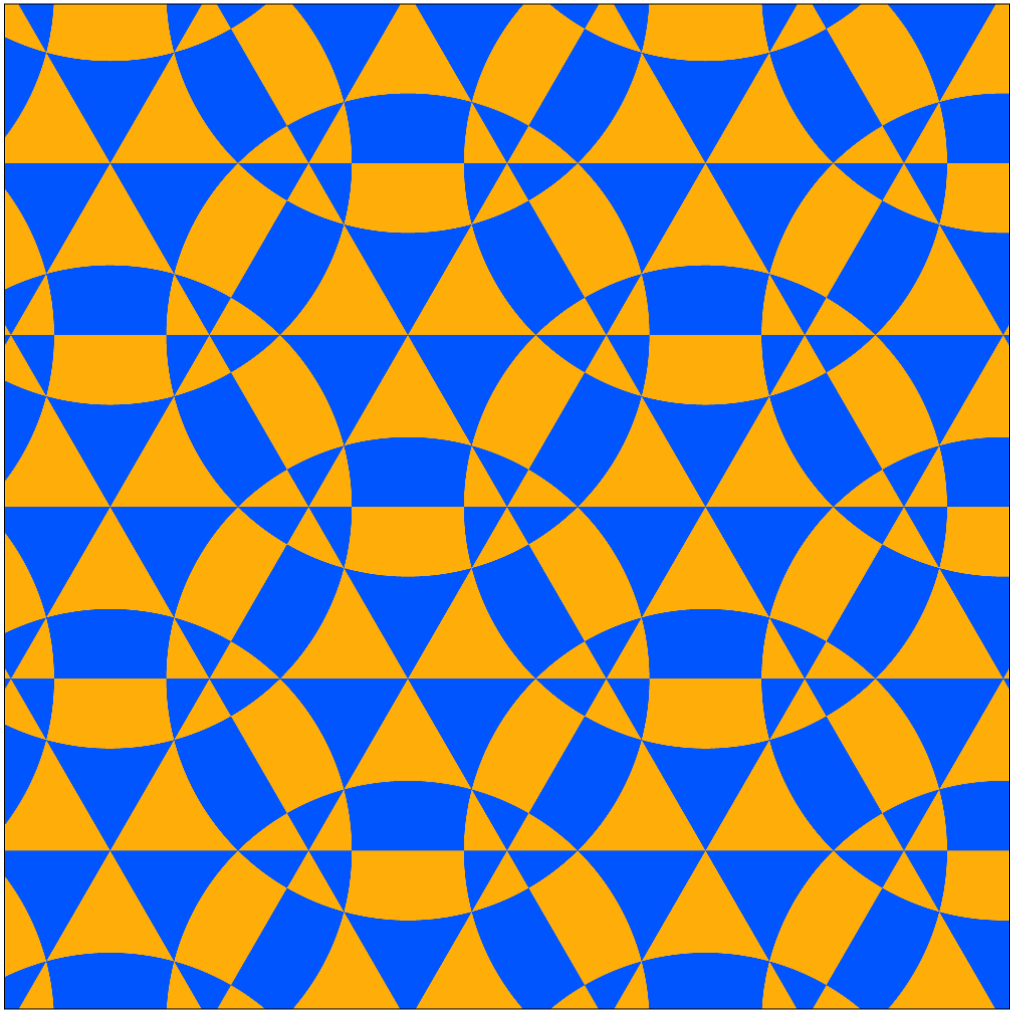
Tetrahedron example 1



This tetrahedron has icosahedral vertex $\star 532$ and ideal vertex $\star 333$.

Animation.
https://youtu.be/AVnNs3dU3-c

Tetrahedron example 2



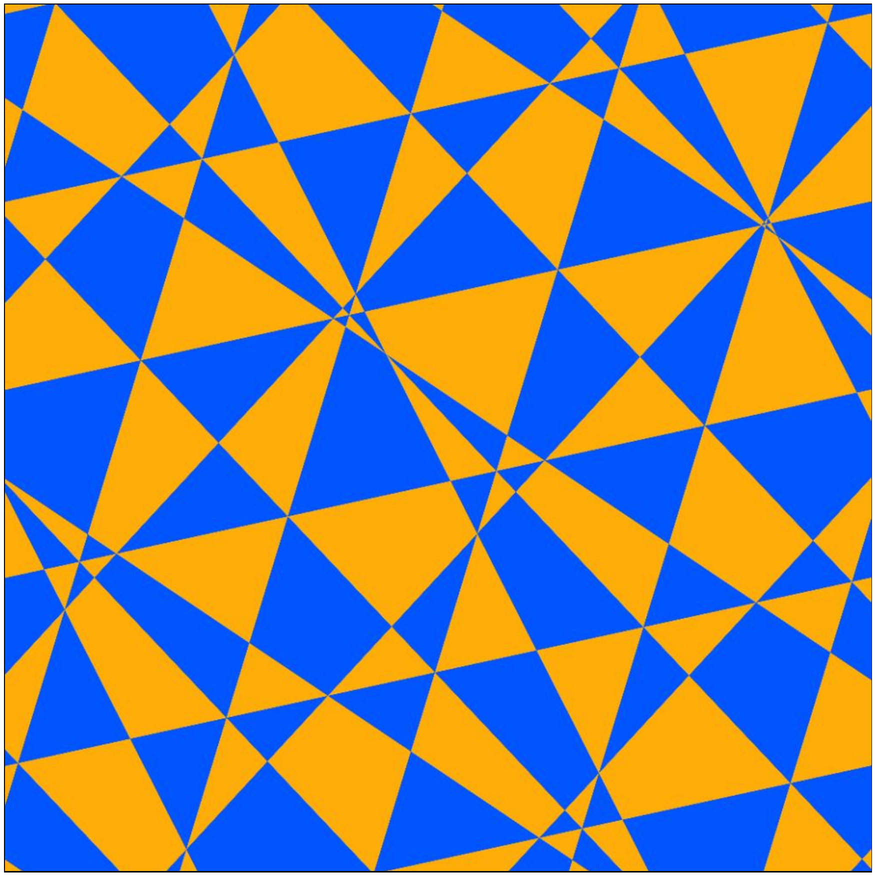
Tetrahedron has ideal vertex $\star 333$.

Horosphere is moving away from that vertex.

Animation.
https://youtu.be/F1cDHtS6oeE

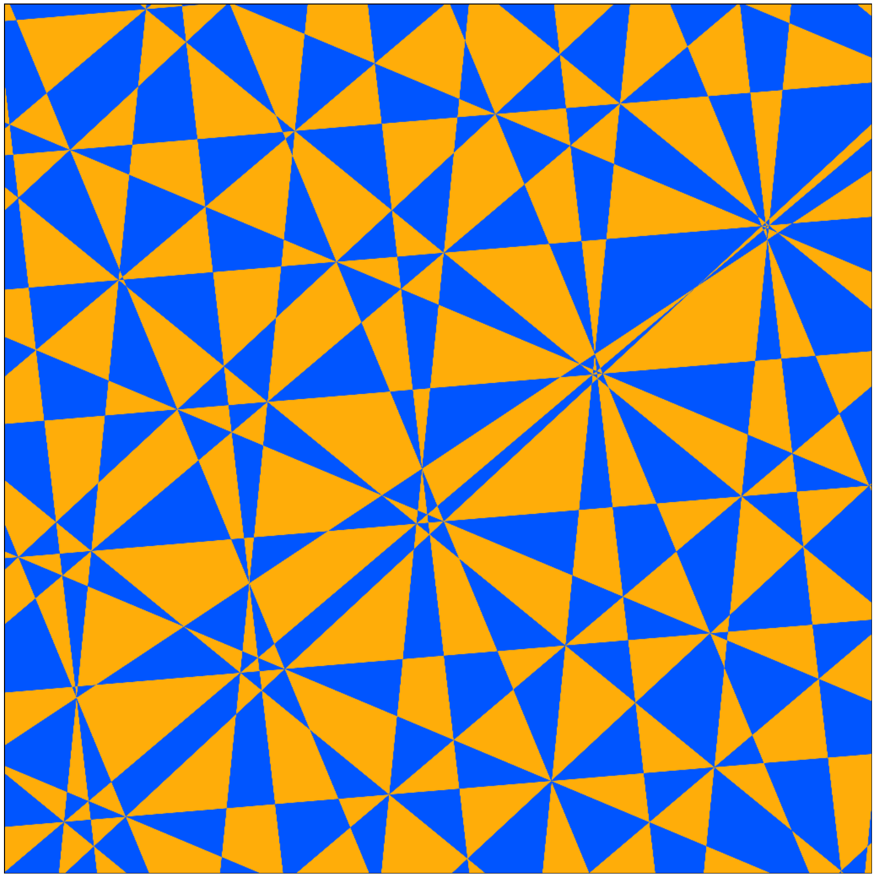
Other Cross Sections

Slicing euclidean tiling wih plane also produces interesting results



Animation 1

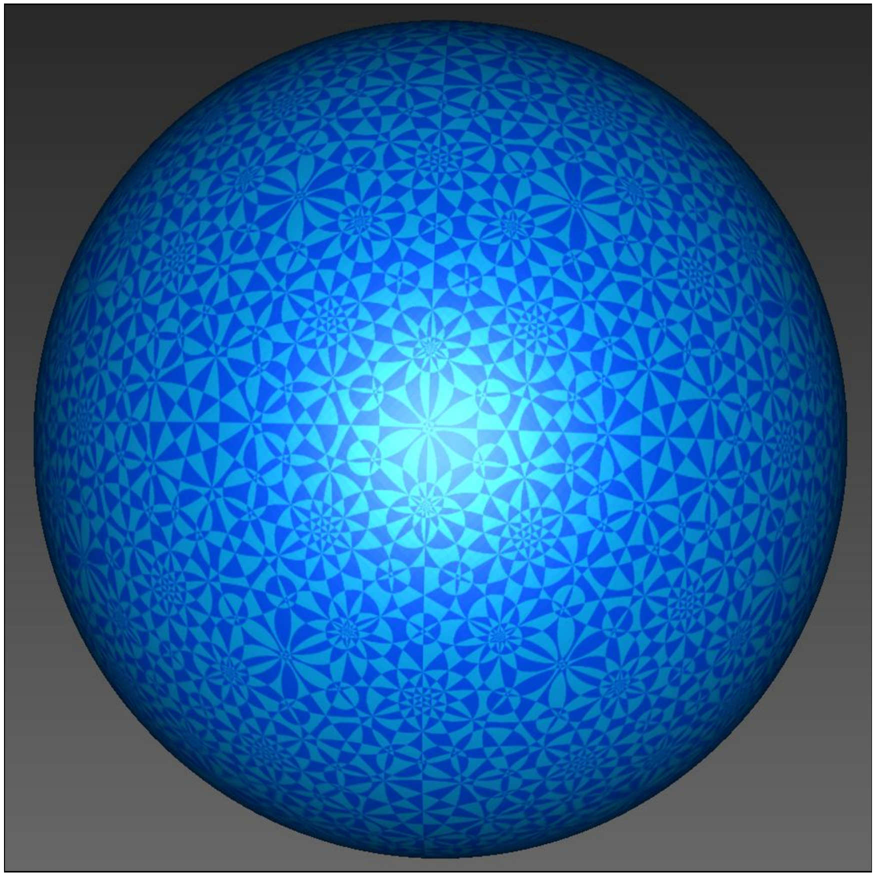
https://youtu.be/tR_IFchXvo8



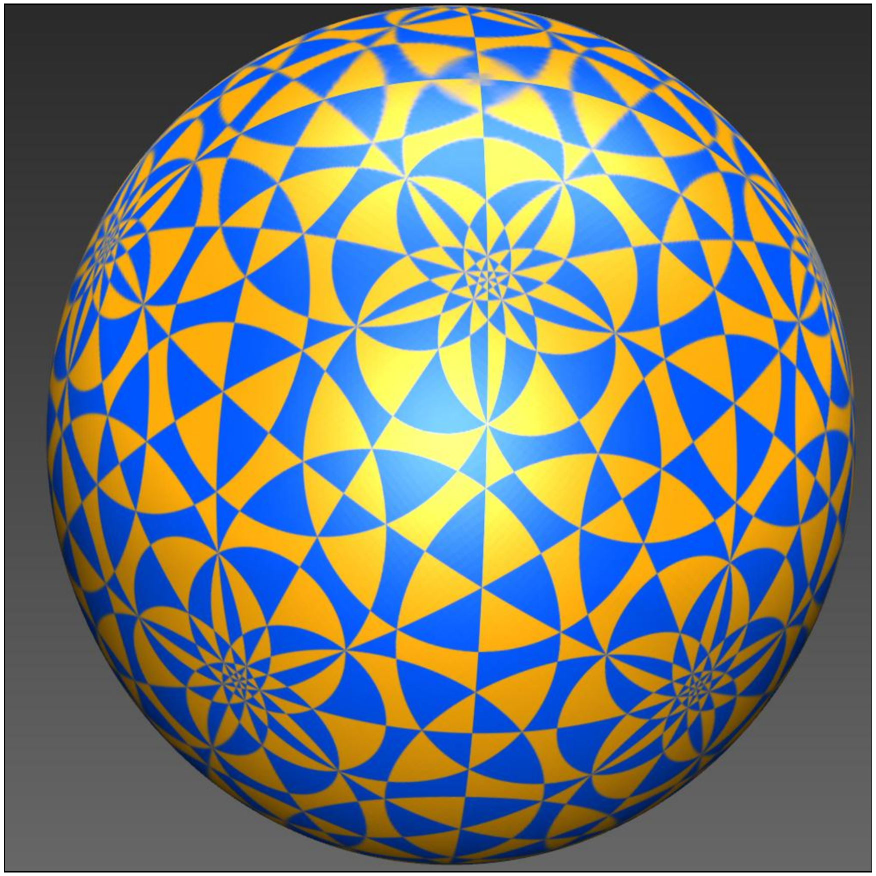
Animation 2

https://youtu.be/-vzEcbNrFnk

Spherical Cross Sections

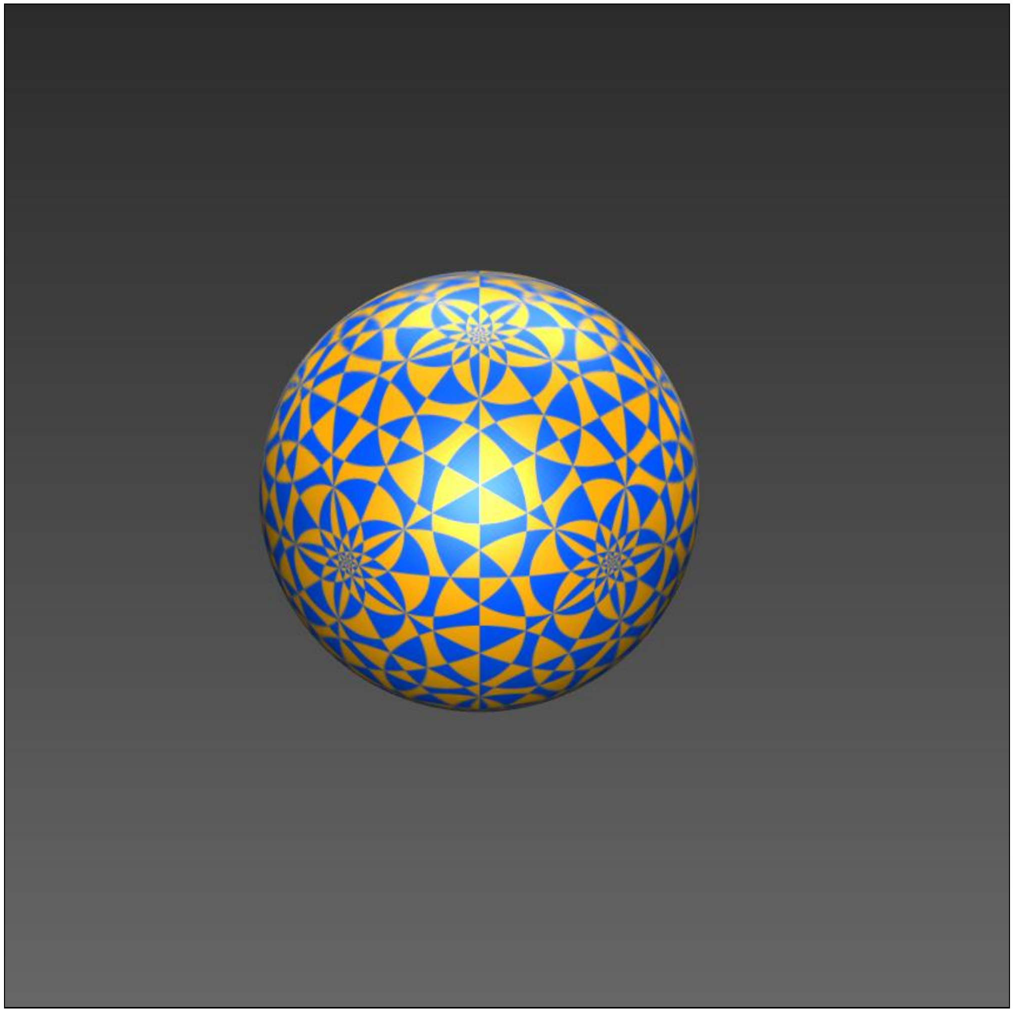


VRML model.



VRML model.

3D Printing

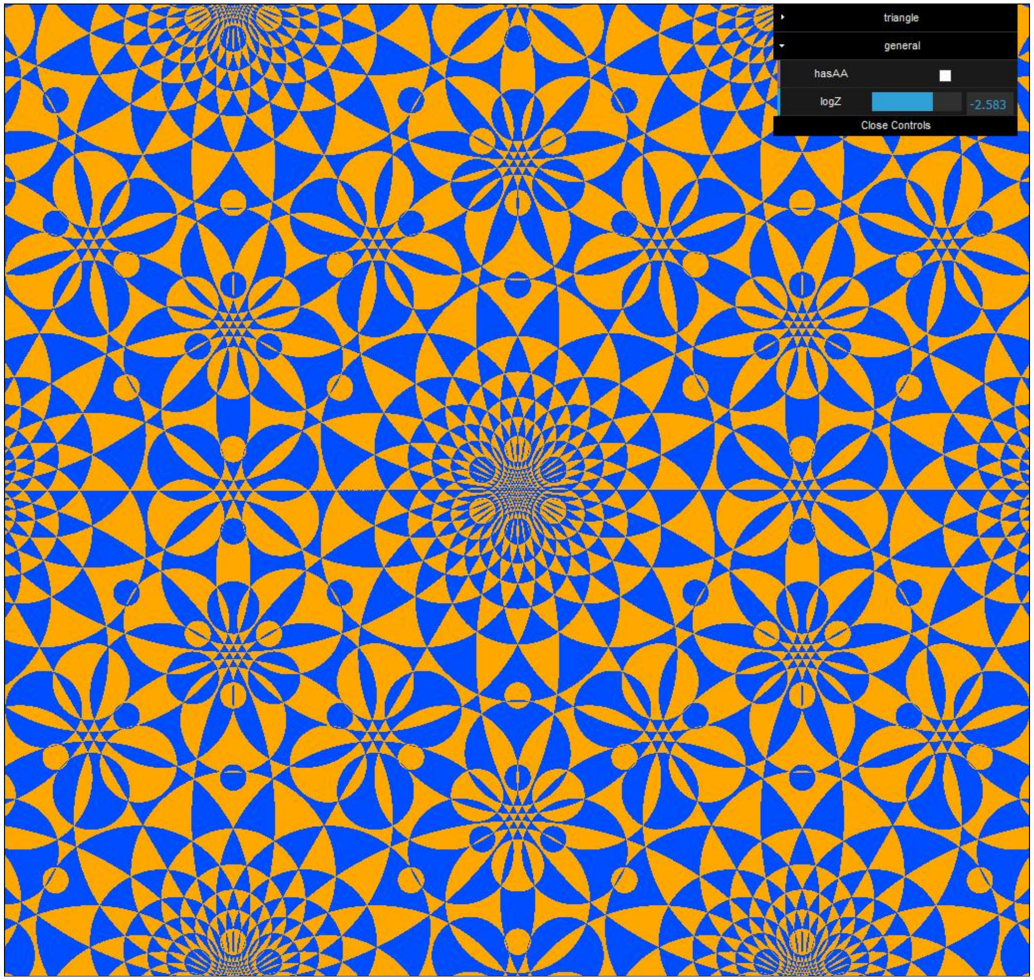


3D printing involves creation of spherical color map and applying it to a shape (sphere).

The modeling was done in **ShapeJS** - online tool for making 3d printable models via script.

The 3D printing was done by Shapeways.

Real Time Rendering with WebGL



hasAA □

logZ -2.583

WebGL is web browser API which allow web pages run graphics program directly on users GPU (graphics card).

WebGL example

End

Online adress of the talk
http://bulatov.org/math/180110/